Using Matrices to Count: Binet's Formula and Characteristic Polynomials

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June 12, 2017

1 Introduction

Today, we'll be looking at how to use matrices to solve certain counting problems, including how to use matrices to find a formula for Fibonacci numbers! To get the most out of this handout, you should be fairly comfortable with matrices. This will be fairly low on explanations and hints as I had to make it fast, so come to me if you get stuck.

2 Eigenvectors

Let M be a two by two matrix. We say that a vector $\mathbf{v} \neq \mathbf{0}$ is an *eigenvector* of M if and only if

$$M\mathbf{v} = \lambda\mathbf{v},$$

for some constant λ. λ is called an *eigenvalue* of M. We can similarly define eigenvectors and values for higher dimensional matrices, but for this paper we'll just need two by two matrices.

Why on earth does anybody care about eigenvectors and eigenvalues? We'll see soon! But first, some practice:

Problem 1. Find the eigenvectors, and their corresponding eigenvalues, of the matrices

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

As you've seen, it's fairly simple to find eigenvalues using systems of equations. But those are long and tedious. Is there any way to generate eigenvalues without systems of equations?

Well, let's look back at the definition $M\mathbf{v} = \lambda\mathbf{v}$. Note that $\lambda(I\mathbf{v}) = \lambda\mathbf{v}$, where I is the identity matrix. So, we're now solving

$$M\mathbf{v} = \lambda(I\mathbf{v}),$$

$$(M - \lambda I)\mathbf{v} = \mathbf{0}.$$

Lemma. The above implies that $\det(M - \lambda I) = 0$.

Proof of Lemma. Use contradiction to show $M - \lambda I$ is not invertible as $\mathbf{v} \neq \mathbf{0}$, then the claim follows.

Using this, we start to get an idea. Can we go backwards? That is, show that if $\det(M - \lambda I) = 0$, then λ is an eigenvalue?

Other Lemma. $\det X = 0$ implies there exists some $\mathbf{v} \neq \mathbf{0}$ such that $X\mathbf{v} = \mathbf{0}$.

Proof of Other Lemma. Let $\mathbf{v}, \mathbf{w}$ be two vectors such that $\mathbf{v} \neq \mathbf{w}$ and $X\mathbf{v} = X\mathbf{w}$ (why can we always pick two such vectors if $\det X = 0$?). Then, $X(\mathbf{v} - \mathbf{w}) = \mathbf{0}$, and $\mathbf{v} - \mathbf{w} \neq \mathbf{0}$.

So yes, we can go backwards.

3 Characteristic Polynomials

We call the polynomial $\det(M - xI)$ the *characteristic polynomial* of matrix M. The roots of the characteristic polynomial are the eigenvalues of M.

Problem 2. Find the eigenvalues of matrices M and N from problem 1 using characteristic polynomials.

Problem 3. (hard) Find a quick way to compute $M^n\mathbf{v}$, given that $\mathbf{v} = a\mathbf{e_1} + b\mathbf{e_2}$, and $\mathbf{e_1}$ and $\mathbf{e_2}$ are two distinct eigenvectors (that are linearly independent) of M with eigenvalues λ_1 and λ_2, respectively.

4 Fibonacci Numbers

Fibonacci numbers are the sequence of numbers defined by the recurrence $F_0 = F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$.

Problem 4. (hard) Find a closed formula for the Fibonacci numbers using characteristic polynomials. Hint: Consider a two by two matrix M such that

$$M \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}.$$

We can think of M as a 'generator matrix' of the Fibonacci numbers.

This formula is usually called Binet's Formula.

5 More Recurrence Problems.

' These might take awhile, but you'll have all summer to work on them!

Problem 5. Find a closed form for the recurrence given by $a_0 = 2, a_1 = 4$, $a_n = a_{n-1} - 2a_n$.

Problem 6. Find a closed form for the recurrence given by $a_0 = 2, a_1 = 4, a_2 = 8$, and $a_n = 3a_{n-3} + 2a_{n-2} + a_{n-1}$.

Problem 7. The formula for Fibonacci numbers uncovered in the last section is pretty sucky. There's a cooler formula using the floor function. Try to find it.

Problem 8. Try finding a matrix M whose eigenvectors are all linearly dependent. What kind of interesting properties does it have? Can you find any recurrences that relate to it?

6 Conclusion

Enjoy your summers! Learning about this in high school was what made me want to major in math, so I really hope this interested you.