A10431W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A3: QUANTUM PHYSICS

TRINITY TERM 2016

Friday, 17 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a fresh page.
For Section B start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Starting from the commutator relation $[\hat{L}_z, \hat{z}] = 0$, derive a relation for m and m' which allows you to evaluate which matrix elements $\langle nlm| \, z \, |nl'm'\rangle$ are non-zero ($|nlm\rangle$ denotes the stationary states of the hydrogen gross structure with the usual quantum numbers). [3]

2. Which quantum numbers define the radial wavefunctions in the gross structure of the hydrogen atom? Sketch the radial wavefunctions for $n = 2$ as a function of r. What is the asymptotic behaviour for $r \to 0$ and $r \to \infty$? How many nodes do the wavefunctions have? Indicate a length scale. [5]

3. Define the probability current for a wavefunction. Show that in one dimension the probability current associated with the wavefunction $\Psi(x) = A\mathrm{e}^{\mathrm{i}kx}$ $(-\infty < x < \infty)$ is $j = |A|^2 \, \hbar k/m$. Sketch as a function of x the wavefunction and the probability density to find the particle between x and $x + \mathrm{d}x$. What property of the wavefunction correlates with the sign of the probability current? [6]

4. A particle of mass m is confined in a 1-dimensional infinite square well potential $V(x) = 0$ for $0 \leq x \leq a$, $V(x) = \infty$ otherwise. At $t = 0$ its normalized wave function is

$$\Psi(x, 0) = \sqrt{\frac{8}{5a}} \left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \sin\left(\frac{\pi x}{a}\right).$$

What is the wavefunction at a later time $t = t_0$?

What is the expectation value of the energy of the system at $t = 0$ and at $t = t_0$?

What is the probability that the particle is found in the left half of the box (i.e. in the region $0 \leq x \leq a/2$) at $t = t_0$? [9]

[Some integrals you might find useful:
$\int_0^{a/2} \sin^2\left(\frac{\pi x}{a}\right) \mathrm{d}x = a/4$, $\int_0^{a/2} \sin^2\left(\frac{2\pi x}{a}\right) \mathrm{d}x = a/4$, $\int_0^{a/2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \mathrm{d}x = 2a/3\pi$.]

5. A particle of mass m moves in a 1-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. In the non-relativistic limit, where the kinetic energy T and momentum p are related by $T = p^2/2m$, the ground state energy is well known to be $\frac{1}{2}\hbar\omega$.

The relativistic kinetic energy is $T = \sqrt{m_0^2 c^4 + p^2 c^2} - m_0 c^2$. Compute the ground state level shift ΔE_0 proportional to c^{-2} (c being the speed of light). [6]

[Hint: You will need to evaluate a sum of expectation values of products of ladder operators. What is $\langle 0|a^\dagger$ and $a|0\rangle$? Which of the expectation values therefore remain?]

$$\left[\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}\right) \text{ and } \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}\right).\right]$$

6. An operator $\hat{f}$ describing the interaction of two spin-$\frac{1}{2}$ particles has the form

$$\hat{f} = a + b\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2,$$

where a and b are constants, and $\hat{\boldsymbol{s}}_1$ and $\hat{\boldsymbol{s}}_2$ are the angular momentum vectors for the two particles. The total spin angular momentum is $\hat{\mathbf{S}} = \hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2$.

Show that $\hat{f}$, $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ can be simultaneously certain. [3]

Derive the matrix representations for $\hat{f}$ in the $|s_1, s_2, S, M_S\rangle$ and in the $|s_1, s_2, m_{s1}, m_{s2}\rangle$ bases (label rows and columns of your matrices). [8]

Section B

7. The total spin of a system of particles with spin $\hat{\mathbf{s}}_\mathbf{i}$ is given by $\hat{\mathbf{S}} = \sum_i \hat{\mathbf{s}}_\mathbf{i}$.

How many orthogonal spin states can a system of two electrons have? State the possible values of the total spin angular momentum quantum number, S, and the possible values for the quantum number, M_S, for each of them. [4]

Find expressions for $\hat{S}^2$ and $\hat{S}_z$ in terms of $\hat{s}_i^2$, $\hat{s}_{z_i}$ and $\hat{s}_i^{\pm}$ with $i = 1, 2$ denoting the two particles. Show that $\hat{S}^2$ and $\hat{S}_z$ commute with the exchange operator P_{12}, which exchanges the coordinates of any two particles 1 and 2. [6]

One of the possible states of a 2-electron system is

$$\Psi = \frac{1}{\sqrt{2}} \left(|\downarrow_1 \uparrow_2\rangle + |\uparrow_1 \downarrow_2\rangle \right).$$

Determine the quantum numbers S and M_S of this state using the operators derived above. What is the exchange symmetry of this state? Why is a state like $|\downarrow_1 \uparrow_2\rangle$ not suitable to describe a system of two indistiguishable particles? [6]

If the system is in the state $|S = 1, M_S = 0\rangle$ and a measurement of the spin of one of the electrons is made, what is the probability of finding this electron in a spin-up state? What is the state after the measurement? Does this state have a defined exchange symmetry? Why is this possible? [4]

[You may assume $\hat{s}_{\pm} |s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}\hbar\, |s, m_s \pm 1\rangle$.]

8. The ladder operators for angular momentum are defined as $\hat{L}_\pm \equiv \hat{L}_x \pm \mathrm{i}\hat{L}_y$. What are the hermitian conjugates for $\hat{L}_+$ and $\hat{L}_-$? Find the commutation relations of $\hat{L}_\pm$ with $\hat{L}^2$ and $\hat{L}_z$. [4]

The normalized eigenstates for $\hat{L}^2$ and $\hat{L}_z$ are the spherical harmonics $Y_l^m(\theta, \phi)$. What are the corresponding eigenvalues? Find the eigenvalue equations of $\hat{L}^2$ and $\hat{L}_z$ for the states $\hat{L}_\pm Y_l^m(\theta, \phi)$. [4]

What is $\hat{L}_+ Y_l^l(\theta, \phi)$ (without calculation)? Use this result, together with the eigenvalue equation for $\hat{L}_z$, to determine $Y_l^l(\theta, \phi)$, up to a normalization constant. [9]

Determine the normalization constant by direct integration. Outline how you would obtain the spherical harmonics with $m < l$ from this result. [3]

$$[\hat{L}_x = \frac{\hbar}{\mathrm{i}}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right), \hat{L}_y = \frac{\hbar}{\mathrm{i}}\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right), \hat{L}_z = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\phi},$$

$$\int_0^\pi \sin^{2l+1}\theta \mathrm{d}\theta = \frac{2\left(2^l l!\right)^2}{(2l+1)!}]$$

9. A quarkonium is a system consisting of a heavy quark of mass m_q, bound to its antiquark, also of mass m_q. The inter-quark potential can be described by

$$V(r) = -\frac{a}{r} + br$$

where a and b are constants and r is the antiquark separation. For this problem you may ignore the spin of the quarks. Given the Bohr formula for the energy levels of the electron in hydrogen

$$E_n^{(0)} = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2},$$

where m is the reduced mass of the electron-proton system, deduce an expression for the energy levels of quarkonium in the approximation that you neglect the second term in $V(r)$ (i.e. $b = 0$). [3]

What are the corresponding degeneracies of the lowest two energy levels ($n = 1$ and $n = 2$)? [2]

Use first-order perturbation theory to calculate the corrections to the lowest two energy levels for $b \neq 0$. [12]

Why is it not necessary to use degenerate perturbation theory for this problem? In this model, which quantum numbers are required to identify the energy of the bound quark-antiquark state? [3]

[You may assume that the wavefunctions for the electron in hydrogen are:

$$u_{100} = R_{10}Y_0^0 = \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \times \frac{1}{\sqrt{4\pi}}$$

$$u_{200} = R_{20}Y_0^0 = \frac{2}{\sqrt{8a_0^3}}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \times \frac{1}{\sqrt{4\pi}}$$

$$u_{21-1} = R_{21}Y_1^{-1} = \frac{1}{\sqrt{24a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \times \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}$$

$$u_{210} = R_{21}Y_1^0 = \frac{1}{\sqrt{24a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \times \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$u_{211} = R_{21}Y_1^1 = \frac{1}{\sqrt{24a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \times \left(-\sqrt{\frac{3}{8\pi}}\right) \sin\theta e^{i\phi}$$

where the Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2$, and that

$$\int_0^\infty e^{-kr} r^n \mathrm{d}r = \frac{n!}{k^{n+1}}, n > -1.]$$

10. The Hamiltonian which describes the interation of an electron (spin-1/2) with an external magnetic field $\boldsymbol{B}$ is

$$\hat{H}_0 = -\boldsymbol{\mu} \cdot \boldsymbol{B} = 2\frac{\mu_{\mathrm{B}}}{\hbar}\boldsymbol{S} \cdot \boldsymbol{B}.$$

Show that, in the case of a static uniform magnetic field, B_0, in the z-direction, the energy eigenvalues are $\pm\mu_{\mathrm{B}}B_0$ and find the energy eigenstates. [3]

Now consider superimposing on the static field $B_0\hat{z}$ a time-dependent magnetic field of constant magnitude B_1, which is rotating in the x-y plane with constant angular frequency ω:

$$B_1(t) = \begin{pmatrix} B_1 \cos\omega t \\ B_1 \sin\omega t \\ 0 \end{pmatrix}.$$

Given that the Hamiltonian is written as $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$, write down a matrix representation of $\hat{V}(t)$ in the basis of eigenvectors for the static field. [2]

A general spin state can be written in this basis as

$$\Psi(t) = c_1(t)\mathrm{e}^{-\mathrm{i}\mu_{\mathrm{B}}B_0t/\hbar}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(t)\mathrm{e}^{\mathrm{i}\mu_{\mathrm{B}}B_0t/\hbar}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Use $\omega_0 = 2\mu_{\mathrm{B}}B_0/\hbar$ and $\gamma = \mu_{\mathrm{B}}B_1/\hbar$, and derive without approximation the coupled differential equations for the amplitudes $c_1(t)$ and $c_2(t)$. [6]

Use $\Omega = (\omega - \omega_0)/2$ and show that this leads to

$$\ddot{c}_2(t) - 2\mathrm{i}\Omega\dot{c}_2(t) + \gamma^2 c_2(t) = 0$$

and solve this differential equation for $c_2(t)$. As this is the solution of a second order differential equation, the solution contains two integration constants. Describe how these can be obtained in the case that the system was in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$. [6]

In this case the solution is

$$c_2(t) = \frac{1}{\sqrt{1 + \Omega^2/\gamma^2}}\mathrm{e}^{\mathrm{i}\Omega t}\sin\sqrt{\Omega^2 + \gamma^2}\,t.$$

Sketch the probability for the system of being in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a function of time for (a) $\omega = \omega_0$ and (b) $\omega \neq \omega_0$. [3]

$$\left[\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$