Math 241 Practice Final August 11, 2017

Name: Answer Key

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

No calculators or notes allowed.

Show all work clearly.

Use proper notation.

Question	Points	Score
1	5	
2	10	
3	5	
4	15	
5	15	
6	10	
7	20	
8	5	
Total	50	

1. (5 points) Find the average rate of change of the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$.

$$m = \frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}$$

2. (10 points) Evaluate the following limits:

   (a) (5 points)

$$\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 2x - 3}$$

$$\frac{3^2 - 9}{3^2 - 3 \cdot 2 - 3} = \frac{9 - 9}{9 - 6 - 3} = \frac{0}{0}$$ und, so can't just plug in

implies $x - 3$ is a factor of top & bottom, try factoring.

$$\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 2x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x+1)} = \lim_{x \to 3} \frac{x+3}{x+1} = \frac{6}{4}$$

$$= \frac{3}{2}$$

(b) (5 points)

$$\lim_{x\to\infty} \frac{cos^2(x)}{x}$$

hint: sandwich theorem

$$\cos(x) \leq 1 \implies \underbrace{\cos^2(x)}_{\text{always positive}} \leq 1 \rightarrow 0 \leq \cos^2(x)$$

$$0 \leq \cos^2 x \leq 1 \implies \cancel{\frac{0}{x}}^{\,0} \leq \frac{\cos^2 x}{x} \leq \frac{1}{x}$$

$$\implies \underbrace{\lim_{x\to\infty} 0}_{=0} \leq \lim_{x\to\infty} \frac{\cos^2 x}{x} \leq \lim_{x\to\infty} \frac{1}{x} = 0$$

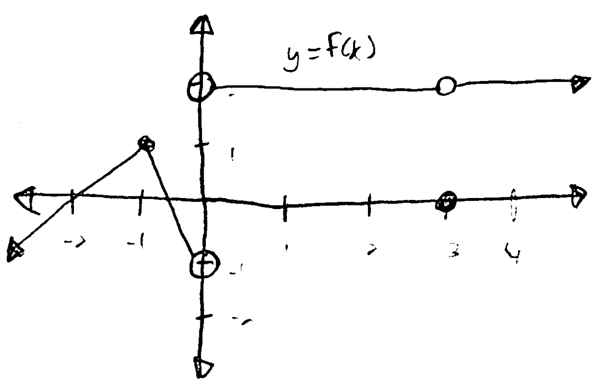
$$\implies 0 \leq \lim_{x\to\infty} \frac{\cos^2 x}{x} \leq 0 \implies \lim_{x\to\infty} \frac{\cos^2 x}{x} = 0$$

3. (5 points) For the given graph, state:

a.) Any values x_0 where $lim_{x\to x_0} f(x)$ does not exist. $X_0 = 0$

b.) The interval(s) of continuity of f $(-\infty, 0), (0, 3), (3, \infty)$

c.) The interval(s) of differentiability of f. $(-\infty, -1), (-1, 0), (0, 3), (3, \infty)$



4. (15 points) Graph $f(x) = x^3 - 3x + 4$, including correct coordinates for any intercepts, local extrema, and inflection points.

my bad, x-intercept too hard for this one!

C.p.'s $f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$

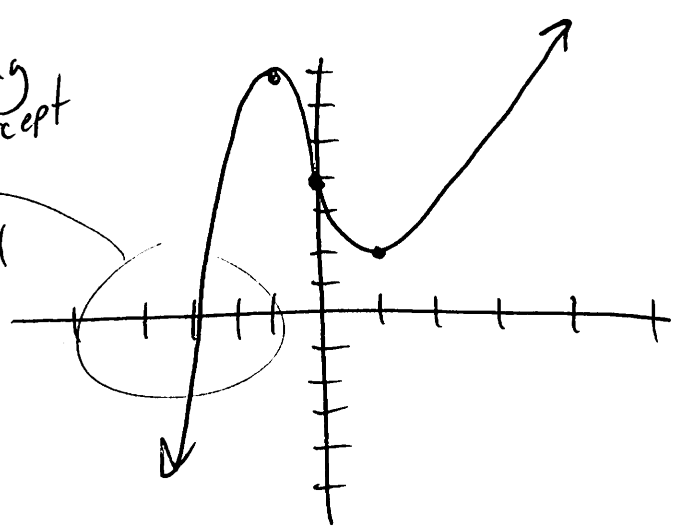
$x=1$, $x=-1$

$f(1) = 1-3+4 = 2$ $f(-1) = -1+3+4 = 6$

$f''(x) = 6x \longrightarrow f''(1) = 6 > 0$ so $(1,2)$ is a local min

$\longrightarrow f''(-1) = -6 < 0$ so $(-1,6)$ local max

i.p's $f''(x) = 6x \rightarrow x=0$. $f(0) = 4$ so $(0,4)$ inflection point.



not actually sure where intercept is b/c 3rd degree polynomial factoring is too hard

Not covered in class

5. (15 points) Differentiate the following functions.

(a) (5 points)

$$q(w) = sin(w)cos(w)$$

$$q'(w) = \frac{d}{dw}(\sin w)\cos w + \sin(w)\left(\frac{d}{dw}\cos w\right)$$

$$= \cos^2 w - \sin^2 w$$

(b) (5 points)

$$r(t) = \frac{t + cos(t)}{\sqrt[3]{t}}$$

$$r'(t) = \frac{\frac{d}{dt}(t+\cos t)\sqrt[3]{t} - \frac{d}{dt}(\sqrt[3]{t})(t+\cos t)}{(t^{1/3})^2}$$

$$= \frac{(1-\sin t)\sqrt[3]{t} - \frac{1}{3}t^{-2/3}(t+\cos t)}{t^{2/3}}$$

or some equivalent expression

(c) (5 points)

$$p(a) = tan(a^2 + sin(a))$$

$$p'(a) = \sec^2(a^2 + \sin(a))\left[\frac{d}{da}(a^2 + \sin(a))\right]$$

$$= \sec^2(a^2 + \sin(a))(2a + \cos(a))$$

6. (10 points) Evaluate the following:

(a) (5 points) $\int_0^{\pi}(\sqrt{x} + cos(x))\, dx$

$$\int_0^{\pi} x^{1/2} + \cos(x)\,dx = \frac{2x^{3/2}}{3} + \sin x \Big|_0^{\pi}$$

$$= \frac{2}{3}\pi^{3/2} + 0 - \left(\frac{2}{3}0^{3/2} + 0\right)$$

$$= \frac{2}{3}\pi^{3/2}$$

(b) (5 points) $\int_0^{\pi/2} cos^2(\theta) sin(\theta)\ d\theta$

let $u = \cos\theta$ $\qquad \frac{du}{d\theta} = -\sin\theta$

$d\theta = \frac{du}{-\sin\theta}$

$0 \longmapsto \cos(0) = 1$

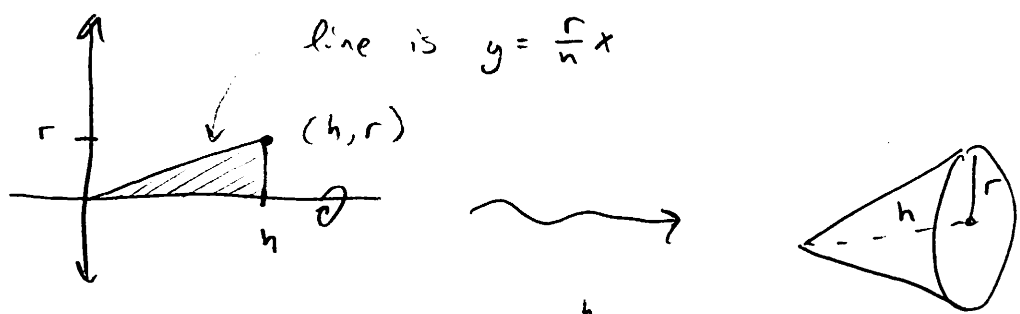
$\pi/2 \longmapsto \cos(\pi/2) = 0$

$$= -\int_1^0 u^2\, du = \int_0^1 u^2\, du$$

$$= \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$

7. (20 points) Note: each part of this problem is possible without the other.

   (a) (10 points) Prove that the volume of a cone is $V = \frac{\pi}{3}r^2h$. The accompanying figure may be useful.



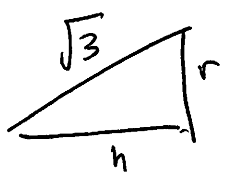
$$V = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi\frac{r^2}{h^2}\int_0^h x^2\,dx$$

$$= \pi\frac{r^2}{h^2}\left(\frac{x^3}{3}\Big|_0^h\right) = \pi\frac{r^2}{h^2}\left(\frac{h^3}{3} - 0\right)$$

$$= \frac{\pi r^2 h}{3}$$

Wow!

(b) (10 points) A cone is generated by revolving a right triangle along one of its legs. Given that the hypotenuse of this triangle is the constant $\sqrt{3}$, what dimensions of the other two legs maximize the volume of the cone?



$$\longrightarrow h^2 + r^2 = (\sqrt{3})^2 = 3$$

$$\longrightarrow r^2 = 3 - h^2$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3 - h^2)h = \pi h - \frac{\pi}{3}h^3$$

maximize V: $V'(h) = \pi - \pi h^2 = \pi(1 - h^2)$

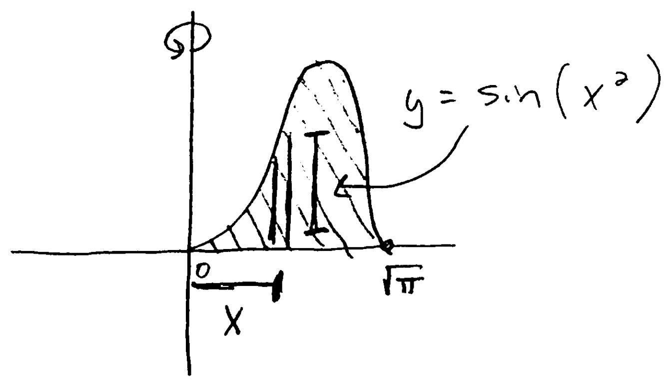
$$V'(h) = 0 \implies h = \pm 1$$

$-1 = h$ makes no sense, so $h = 1$

$$h = 1 \implies r = \sqrt{3 - h^2} = \sqrt{3 - 1} = \sqrt{2}$$

So $h = 1$, $r = \sqrt{2}$ maximize the volume of the cone.

8. (5 points) Find the volume of the solid generated by revolving the shaded region about the y-axis.



Can't use washer method: both radii are from same curve.

$\Rightarrow$ shell method.

$$V = 2\pi \int_0^{\sqrt{\pi}} \sin(x^2)\, x \, dx$$

let $u = x^2$ $\quad du = 2x\, dx \quad dx = \frac{du}{2x}$

$0 \mapsto 0^2 = 0$, $\sqrt{\pi} \mapsto \sqrt{\pi}^2 = \pi$

$$V = 2\pi \int_0^{\pi} \sin u \, x \frac{du}{2x} = \pi \int_0^{\pi} \sin u \, du$$

$$= \pi \left[-\cos u\right]\Big|_0^{\pi} = \pi\left[-\cos \pi - (-\cos 0)\right]$$

$$= \pi[1 + 1] = 2\pi$$