THE TUTTE-GROTHENDIECK RING

T. H. BRYLAWSKI

1. Introduction

The history of combinatorial theory has been marked by ingenious, albeit ad hoc, methods. It is perhaps the inherent discrete and unstructured nature of most combinatorial models that has placed combinatorics out of the mainstream of current mathematical research. Unfortunately, while many mathematicians are content to generalize rigid axiomatic structures (e.g. modules), the combinatorialist has heretofore been obliged to make do with the lattice, the graph, or the partial order. The recent work of G.-C. Rota and collaborators (H. Crapo, C. Greene, R. Stanley, *et al.*) has been marked by an attempt to coordinate and regiment combinatorial theory as a more classical mathematical science. They have borrowed freely from the methods of other branches of mathematics, especially algebra.

The theory of commutative algebras and modules has led to the study of abelian categories and the Grothendieck ring (and group) which possess all the information which is contained in the classical matrix case in the notion of trace and determinant. It was found by the author that such algebraic constructions, appropriately generalized lent much insight into the theory of combinatorial geometries [3] and hence a re-axiomatization of these notions might allow other mathematicians to profitably apply these constructive techniques to other fields.

We add at the outset, however that this ring construction was first intuited in a profound paper by W. Tutte in 1947 [14], therefore, credit for the idea of a Grothendieck group and ring belongs in part to Tutte.

Motivated by the way the classical Grothendieck group arises from the invariance of the trace of a linear operator on certain vector space decompositions we are led to define and explore a *decomposition category* which allows us to exploit these Tutte-Grothendieck methods for the study of general decompositions or bidecompositions (two decompositions on the same set) and their invariant functions.

In studying these two decompositions (one reminiscent of direct sum or product and the other of subobject-quotient object decomposition) one notes that it is precisely when these techniques are applied to combinatorial structures that one observes a uniqueness property for each decomposition and a compatibility between them resembling the distributive compatibility of ring operations. One can then construct a free commutative ring called the Tutte-Grothendieck ring and canonical map, the Tutte invariant, which composes with ring homomorphisms to give a 1–1 correspondence between such homomorphisms and those functions which, like the charac-

Presented by R. S. Pirece. Received August 24, 1971. Accepted for publication in final form October 2, 1972.

teristic polynomial for combinatorial geometries, are invariant for the above decompositions. In fact it is the freedom of this ring which in a vague sense suggests the combinatorial nature of the decomposition – it is in the examples from algebra that relations are introduced in the ring which obscure an explicit characterization of its structure.

We construct this ring for two reasons. First, it gives us a framework in which one may 'compose' two objects in the ring in a manner adjoint to the decomposition, although no such formal composition may exist in the category. Second, it allows us to compute the Tutte invariant (or Tutte polynomial) which then serves as a universal invariant for the decomposition.

We believe that the present approach, based on a somewhat different idea from Grothendieck's allows a broad range of applications. We first define a decomposition category which encompasses most of the classical examples of decompositions but at the same time is much too general to be of use. One first needs to consider the more well-behaved decompositions – especially the ones which admit common refinement and those in which every element may be decomposed into irreducible (indecomposable) elements.

The theory as it is outlined in this paper bears some resemblance on one side to current work on homological algebra, and on the other side to some older work of Brockway MacMillan. We have not explored the connections with MacMillan's work.

Our main result is essentially a basic condition, which we give in the form of a definition, which must be satisfied by the decomposition category in order that its associated Tutte-Grothendieck ring be a polynomial ring in variables canonically associated with objects which are indecomposable (irreducible) for both decompositions. While we believe that decomposition categories, as introduced here, will have other combinatorial and algebraic applications, and although we present applications to linear algebra, matrix theory, design theory, topology, and group permutation representations; the most fruitful applications so far have been found in the theory of combinatorial geometries ([1], [2], and [3]), finite ordered sets [12], coding theory [7], and generating functions [8].

I would especially like to thank Professor Gian-Carlo Rota who first suggested these ideas [10] and also thank Professors Rota, H. Crapo, R. Davis, L. Geissinger, R. Heyneman, L. Solomon, and the referee for their ideas and suggestions.

2. Decompositions

In this section decompositions are axiomatized and the special case is described when two decompositions on the same set admit certain compatibility relations.

DEFINITION 2.1. A *multiset* (or *multisubset*) of a nonempty set S is a function f from S into the nonnegative integers. If s is an element of $S, f(s)$ is called the *multi-*

plicity of s. A finite multiset is one with support on a finite set. In this case we define the *cardinality* of f, $|f|$, as $\sum_{s\in S} f(s)$. Note that f is empty iff $|f|=0$. Also, if $f(s)\leqslant 1$ for all $s\in S$, f may be considered as a subset of S.

Multisets admit a commutative composition, *disjoint union* given by $f+g$. When restricted to finite multisets, this composition gives a structure naturally isomorphic to $FCM(S)$, the free commutative monoid with S as the set of generators. When no confusion will arise we will identify an element $s\in S$ with its characteristic multiset, $\chi_s=\delta(s,.)$. Hence, if $|f|=n$, a multiset f can be written in the form $\sum_{i=1}^{n} s_i$.

A *decomposition* of a set S is a category $\mathbf{D}(S)$ whose objects are finite nonempty multisets of S (a subset of the free commutative semigroup of S, $FCM^*(S)$) and whose morphisms obey the following axioms:

a. The morphism class $[f, g]$ contains at most one element and is empty if either $|f|>|g|$ or if $|f|=|g|$ but $f\neq g$.

(Note that $\mathbf{D}(S)$ is a partial order on multisubsets of S in which members of S are the minimal elements, hence we will express the fact that the morphism class $[f, g]$ is nonempty by $f\leqslant g$ in which case we say f *decomposes* into g. The above axiom simply states that cardinality is a functor into the ordered natural numbers which takes only identities onto identities.)

b. $f+g\leqslant h$ iff there exist h_1 and h_2 such that $h_1+h_2=h, f\leqslant h_1$, and $g\leqslant h_2$.

DEFINITION 2.2. An element $i\in S$ is *irreducible* if $i\nless f$ for all $f\in\mathbf{D}(S)$. Hence the irreducible elements of S are the incomparable elements. $\mathbf{D}(S)$ is a *finite* decomposition if for each $s\in S$, $s\leqslant I$ for some multiset I of irreducibles. We then say s *fully decomposes* into I. Note that the elements I are just the maximal elements of $\mathbf{D}(S)$ and that chains terminating in I can have length at most $|I|$.

A decomposition is *refinable* if $f\leqslant g$ and $f\leqslant h$ imply there is a multiset termed a *common refinement* of g and h such that $g\leqslant j$ and $h\leqslant j$. Decomposition categories with pushouts are thus refinable.

By (2.1.b) a multiset f cannot be decomposed into any other multiset iff it is a multiset of irreducibles. Hence, if $\mathbf{D}(S)$ is refinable, for any $s\in S$ there is at most one way to decompose s into irreducibles. We are thus justified in calling a finite refinable decomposition *unique*.

A decomposition is *coadditive* if for every finite nonempty multiset f there is some $s\in S$ such that $s\leqslant f$ (as in a cocommutative cogroup). A unique coadditive decomposition in which every multiset I of irreducibles is the codomain of a unique element of S is termed *factorable*.

Note that if S is the set of (equivalence classes of) all partial orders which have a bound on the number of elements in an antichain and if $\mathbf{D}(S)$ is partitioning into subpartial orders with chains (total orders) as irreducibles, finiteness/of $\mathbf{D}(S)$ follows from an infinite analog of a theorem by Dilworth [6].

An important result in lattice-ordered vector spaces concerns a property similar to refinability.

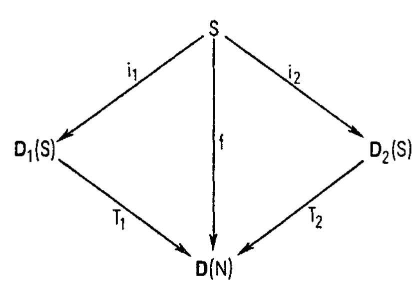
If S consists of all nonempty subsets of a set S' decomposed by (finite) partitioning, then clearly, $\mathbf{D}(S)$ is refinable and is unique if and only if S' is finite.

A *generating set* for $\mathbf{D}(S)$ is a subcollection M of morphisms (with their objects) such that no proper decomposition subcategory contains M. A minimal such generating set is termed a *basis*. Axiom (2.1.b) shows that an example of a basis is the set of all morphisms of the form $s<f$ where $s \in S$ and f covers s in the partial order.

EXAMPLE 2.3. An example of a decomposition which is not refinable occurs when S is the set of isomorphism classes $[s]$ of finite nontrivial groups s and where $\mathbf{D}$ is generated by $[s]<[s_1]+[s_2]$ for all subgroups s_1 and s_2 of s such that $s=s_1s_2$ and $s_1 \cap s_2 = 1$. Then, if D_4 is the dihedral group presented as $\{c, d \mid c^4=d^2=(cd)^2=1\}$, it can be decomposed into the irreducible subgroups $\{1, d\}$ and $\{1, c, c^2, c^3\}$ isomorphic to Z_2 and Z_4 respectively. However, it can also be decomposed into $\{1, d\}$ and $\{1, c^2, cd, c^3d\}$ the latter subgroup being further decomposable into $\{1, cd\}$ and $\{1, c^3d\}$, and hence we have the two decompositions $D_4<Z_2+Z_4$ and $D_4<Z_2+Z_2+Z_2$ with no common refinement.

DEFINITION 2.4. An *integer decomposition* $\mathbf{D}(N)$ is one in which N is some subset of the positive integers and for each morphism $n<\sum_i n_i$, $n_i<n$ (as integers). A decomposition $\mathbf{D}$ is *strictly finite* if there exists a functor T from $\mathbf{D}$ into an integer decomposition which sends nonidentities into nonidentities. It is easy to see that a decomposition is strictly finite if and only if all chains from a given s have bounded length $1(s)$; hence a strictly finite decomposition is finite and a unique decomposition is strictly finite (e.g. we can let $T(s)=|I|$, where I is a full decomposition of s).

DEFINITION 2.5. Let $\mathbf{D}_1(S)$ and $\mathbf{D}_2(S)$ be two strictly finite decompositions on the same set with nonidentity preserving functors T_1 and T_2 mapping $\mathbf{D}_1(S)$ and $\mathbf{D}_2(S)$ into the same integer decomposition $\mathbf{D}(N)$ such that the following diagram is commutative for the canonical injections i_1 and i_2, and some function f:



We then say $\mathbf{D}_1$ and $\mathbf{D}_2$ form a *bidecomposition* $\mathbf{B}(S)$. For clarity we denote multisets $\sum_{i=1}^{n} s_i$ of $\mathbf{D}_1$ as $\bigtimes_{i=1}^{n} s_i$ and call an irreducible element of $\mathbf{D}_1$ an *indecomposable*. (Note that not every two finite decompositions on the same set form a bidecomposition. If $S=\{a, b\}$ and $a<b+b$ while $b<a\times a$ no bidecomposition is possible.)

If $\mathbf{D}_1$ and $\mathbf{D}_2$ are unique and every element of S is either indecomposable or irreducible we say $\mathbf{B}$ is *free*. In this case the union of the $\mathbf{D}_1$ and $\mathbf{D}_2$ morphisms forms a unique, hence strictly finite decomposition. Series-parallel decomposition of finite networks ([1]) is an example of a free decomposition.

DEFINITION 2.6. A bidecomposition is *distributive* if it satisfies the following axioms:

a. $\mathbf{D}_1$ and $\mathbf{D}_2$ are both unique.

b. If $s<\bigtimes_{j=1}^{n} s_j \in \mathbf{D}_1$ for some multiset $\{s_j\}$ of elements which are irreducible (in $\mathbf{D}_2$), then s is irreducible (in $\mathbf{D}_2$).

c. The distributivity axiom: If $s<\bigtimes_{i=1}^{n} f_i \in \mathbf{D}_1(S)$ and $f_k<\sum_{i=1}^{n} h_j \in \mathbf{D}_2(S)$ for some $k\in[1, n]$, then there exist m multisets g_j such that $s<\sum_{j=1}^{m} g_j \in \mathbf{D}_2$ and, for each j, $g_j<\bigtimes_{i=k} f_i \times h_j \in \mathbf{D}_1$.

In the $\mathbf{D}_1$-decomposition $s<\bigtimes_{i=1}^{n} s_i$, the distributivity axiom implies that a nontrivial $\mathbf{D}_2$-decomposition of some s_i induces a nontrivial $\mathbf{D}_2$- decomposition of s. Hence an irreducible (for $\mathbf{D}_2$) fully decomposes in $\mathbf{D}_1$ into a multiset of elements which are both indecomposable and irreducible.

More generally, if a decomposition $\mathbf{D}_1$ satisfies (2.6.c) with respect to a decomposition $\mathbf{D}_2$ we say $\mathbf{D}_1$ *distributes over* $\mathbf{D}_2$.

PROPOSITION 2.7. *If* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are two decompositions which obey* (a), (b), *and* (c) *of* (2.6) *then they form a distributive bidecomposition.*

Proof. We need only find the function f of (2.5) which forms a bidecomposition from $\mathbf{D}_1$ and $\mathbf{D}_2$. For all $s\in S$, let $f(s)=\sum_{i=1}^{n} k_i$ where $s<\sum_{i=1}^{n} s_i$ is a full $\mathbf{D}_2$ decomposition and for each i, let $k_i=2^{m_i}$ where $s_i<\bigtimes_{j=1}^{m_i} s_{ij}$ is a full $\mathbf{D}_1$ decomposition (into irreducible indecomposables). Then by distributivity and uniqueness, f induces a bidecomposition.

DEFINITION 2.8. Let $\mathbf{D}(S)$ be a decomposition on a set S. Then a subset S' of S is *hereditary* for $\mathbf{D}(S)$ if $s\in S'$ and $s<\sum s_i \in \mathbf{D}$ imply $s_i\in S'$ for all i. Similarly S' is hereditary for a bidecomposition $\mathbf{B}(\mathbf{D}_1, \mathbf{D}_2)$ if it is hereditary for $\mathbf{D}_1$ and $\mathbf{D}_2$. If $S'\subseteq S$ is hereditary, the full subcategory $\mathbf{D}(S')$ (where the objects are multisubsets of S') is called a *subdecomposition*. We have a similar notion of a *subbidecomposition*, $\mathbf{B}(S')$.

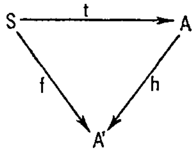
3. The Tutte-Grothendieck group and ring

The study of functions invariant under a decomposition (bidecomposition) is facilitated by the construction of a universal group (ring) associated with the decomposition(s).

DEFINITION 3.1. A *decomposition invariant* (**D**-*invariant*) function f for a decomposition $\mathbf{D}(S)$ is a function with domain S taking values in some abelian group A such that for every morphism $s<\sum_{i=1}^{n} s_i$ in $\mathbf{D}(S)$, $f(s)=\sum_{i=1}^{n} f(s_i)$.

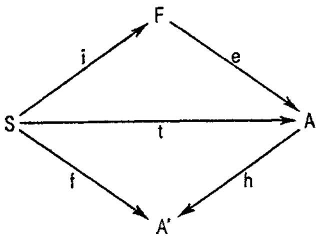
Note: By applying (2.1.b) and addition in A, the above definition is equivalent to requiring equalities for all morphisms in $\mathbf{D}(S)$ or, equivalently, requiring equalities for only those morphisms in a basis.

THEOREM 3.2. *Let* $\mathbf{D}(S)$ *be a unique decomposition. Then there exists an abelian group* A *called the Tutte-Grothendieck group and (invariant) function* t *taking* S *into* A *such that for any abelian group* A' *in the following diagram:*



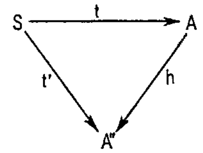
any group homomorphism h *gives rise to a unique* **D**-*invariant function* f *by composition with* t*; and for any* **D**-*invariant* f *there is a unique homomorphism* h *such that the diagram commutes. In addition* A *is isomorphic to the free abelian group whose generators are the irreducible elements of* $\mathbf{D}(S)$.

Proof. Let $F(S)$ denote the free abelian group with generators the elements of S and let $i:S\to F$ be the canonical injection. Further, let T be the functor from $\mathbf{D}(S)$ into F (viewed as a one object category) which sends each morphism $\sum_{i=1}^{n} s_i<\sum_{j=1}^{m} s_j$ into the morphism (group element) $\sum_{i=1}^{n} s_i-\sum_{j=1}^{m} s_j$. Let G be the subgroup of F generated by all elements in the image of T (equivalently, those images under T of any generating set for $\mathbf{D}(S)$), and further let A be the quotient group F/G. Then, if e is the canonical epimorphism $e:F\to A$ which sends an element f in F to its coset $f+G$ we have the following commutative diagram:

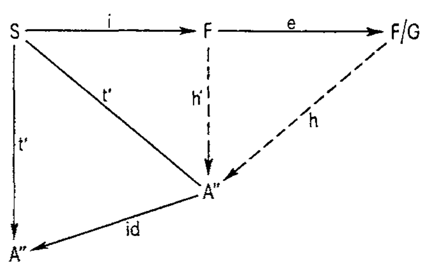


in which the pair (A, t) has the property P that any group homomorphism h when composed with t is a **D**-invariant function (since for any decomposition $s<\sum_i s_i$, $s-\sum_i s_i$ is in the kernel of e and hence $t(s)=\sum_i t(s_i)$ so $(h\circ t)(s)=\sum_i (h\circ t)(s_i)$).

Further, (A, t) is universal in that for any other pair (A'', t') with property P there exists a (unique) homomorphism h such that $t'=h\circ t$ in the following commutative diagram:



Referring to the diagram below



we can construct the homomorphism h' so that $t'=h'\circ i$ since F is free and we need only specify that $h'(s)=t'(s)$ for all generators s of F (i.e. all $s\in S$) and then extend h' linearly. Since any homomorphism into an abelian group when composed with t' forms a **D**-invariant function, we see that t' itself must be **D**-invariant by composing it with the identity on A''. But this means that the kernel of h' contains all the identities which generate G (those which make t' **D**-invariant) and this implies that we may extend the homomorphism h' to h.

Let A'' be the free abelian group with the irreducible elements of $\mathbf{D}(S)$ as generators and let $t':S\to A''$ be such that $t'(s)=\sum_i s_i$ where $s<\sum_i s_i$ is the full decomposition of s in the unique decomposition $\mathbf{D}(S)$. Then t' is a **D**-invariant since if $s<\sum_{j=1}^m s_j$ and if $s_j<\sum_{k=1}^{n_j} s_{jk}$ is a full decomposition of each s_j then by (2.1.b) and composition of morphisms, $s<\sum_{j=1}^m \sum_{k=1}^{n_j} s_{jk}$ is the full unique decomposition of s and hence $t'(s)=\sum_{j=1}^m \sum_{k=1}^{n_j} s_{jk}=\sum_{j=1}^m t'(s_j)$.

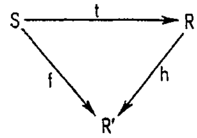
Let h be the homomorphism from A to A'' such that $h\circ t=t'$ which exists by the universality of (A, t). Since for every irreducible i, $t'(i)=i$ and since A'' is free, there can be no nontrivial relations in A among the cosets $[i]$. But for all $s\in S$, G contains an element corresponding to the full decomposition of s; therefore, every generator of F (and hence any element of F) is equal $\operatorname{mod} G$ to an element involving only irreducibles. Thus A is isomorphic to A''.

For any abelian group A' and $\mathbf{D}$-invariant f, let h be the unique homomorphism from the free abelian group A into A' such that $h([i])=f(i)$ for all generators (irreducibles) i of A. Then $f=h\circ t$ and we are done.

DEFINITION 3.3. A *bidecomposition invariant* (**B**-*invariant*) function f for a bidecomposition $\mathbf{B}(S)$ is a function with domain S taking values in some commutative ring R such that for every morphism $s<\bigtimes s_i$ in $\mathbf{D}_1(S)$, $f(s)=\prod f(s_i)$; and for every morphism $s<\sum s_j$ in $\mathbf{D}_2(S)$, $f(s)=\sum f(s_j)$.

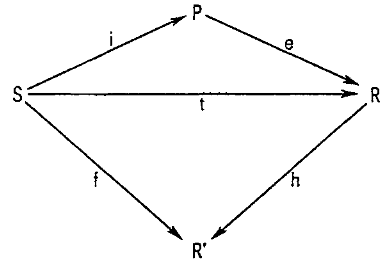
Note that if $\mathbf{B}(S)$ is distributive it is sufficient that $f(s)=\prod f(s_i)$ for every full decomposition $s<\bigtimes s_i$ of a $\mathbf{D}_2$-indecomposable into irreducible indecomposables, since if $s'<\bigtimes s_i'$ is a full $\mathbf{D}_1$-decomposition then one may apply the distributivity axiom to each s_j' to decompose it to a case where for all k, each s_{jk}' is an irreducible indecomposable. The result then follows from the distributivity of R.

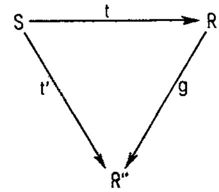
THEOREM 3.4. *Let* $\mathbf{B}(S)$ *be a distributive bidecomposition. Then there exists a commutative ring R called the Tutte-Grothendieck ring and an (invariant) function t from S into R such that for any commutative ring R′ in the following diagram, any*



ring homomorphism h gives rise to a unique **B**-*invariant function f by composition with t and for any* **B**-*invariant f there is a unique homomorphism h such that the above diagram commutes. In addition, R is isomorphic to the free commutative ring (without unit) whose generators are the irreducible indecomposable elements of S.*

Proof. Much of the following is analogous to the proof of (3.2). Let P be a free ring whose generators are the elements of S, and let I be the ideal of P generated by the elements of the form $s-\prod_i s_i$ for every $s<\bigtimes_i s_i$ in $\mathbf{D}_1(S)$ and $s-\sum_j s_j$ for every $s<\sum_j s_j$ in $\mathbf{D}_2(S)$. Then if $R=P/I$ we have the following commutative diagrams:





where the pair (R, t) has the universal property that for any commutative ring R' and ring homomorphism $h: R \to R'$, $h \circ t$ is a **B**-invariant; and for any other pair (R'', t') with this property, then there is a ring homomorphism $g: R \to R''$ such that $t' = g \circ t$.

Let R'' be the free commutative ring generated by the set of irreducible indecomposable elements of S and let $t'(s) = \sum_{j=1}^{n} \prod_{i=1}^{n_j} s_{ij}$ where $s < \sum_{j=1}^{n} s_j$ is a full $\mathbf{D}_2$-decomposition of s; and for each j, $s_j < \bigtimes_{i=1}^{n_j} s_{ij}$ is a full $\mathbf{D}_1$-decomposition of s_j. By the remarks in (2.6) s_{ij} is an irreducible indecomposable for all i and j. We will have proved the theorem as in (3.2) if we show that t' is **B**-invariant. But if $s < \sum_{j=1}^{m} s_j$ is a morphism of $\mathbf{D}_2(S)$ then $t'(s) = \sum_{j=1}^{m} t'(s_j)$ as in (3.2).

Assume $s < \bigtimes_{k=1}^{n} s_k$ is in $\mathbf{D}_1$. For each s_k let $s_k < \sum_{i_k=1}^{n_k} s_{ki_k}$ be a full $\mathbf{D}_2$-decomposition, and for each s_{ki_k}, let

$$s_{ki_k} < \bigtimes_{j=1}^{n_{i_k}} s_{ki_kj}$$

be a full $\mathbf{D}_1$-decomposition. Then,

$$\prod_{k=1}^{n} t(s_k) = \prod_{k=1}^{n} \sum_{i_k=1}^{n_k} \prod_{j=1}^{n_{i_k}} t(s_{ki_kj}) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_n=1}^{n_n} \prod_{k=1}^{n} \prod_{j=1}^{n_{i_k}} t(s_{ki_kj})$$

by distributivity in R''. But by repeated application of the distributivity condition of **B** (2.6) we see that

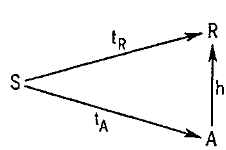
$$s < \sum_{i_1=1}^{n_1} \cdots \sum_{i_n=1}^{n_n} D_{i_1 \ldots i_n}$$

must be a morphism of $\mathbf{D}_2$ and further for each n-tuple $(i_1, \ldots, i_n)$,

$$D_{i_1 \ldots i_n} < \bigtimes_{k=1}^{n} \bigtimes_{j=1}^{n_{i_k}} s_{ki_kj}$$

is a $\mathbf{D}_1$ morphism. But each s_{ki_kj} is an irreducible indecomposable hence by (2.6) each $D_{i_1 \cdots i_n}$ is irreducible and by uniqueness of full decomposition in $\mathbf{D}_1$ and $\mathbf{D}_2$, $t(s) = \prod_{i=1}^{n} t(s_i)$.

PROPOSITION 3.5. *If* $\mathbf{B}(\mathbf{D}_1, \mathbf{D}_2)$ *is a distributive bidecomposition on a set S we have the following commutative diagram:*



where A is the Tutte-Grothedieck group associated with $\mathbf{D}_2$, *R is the Tutte-Grothendieck ring of* $\mathbf{B}$, *and h is a group homomorphism such that* $h(s)=\prod s_i$ *for all* $\mathbf{D}_2$*-indecomposable elements s and full* $\mathbf{D}_1$ *decompositions* $s<\bigtimes s_i$. *In addition, h is an epimorphism if and only if* $\mathbf{D}_1$ *is coadditive, and an isomorphism if and only if* $\mathbf{D}_1$ *is factorable.*

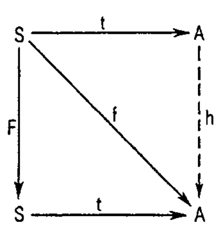
Proof. By (2.6) the function h' defined by $h'(s)=\prod s_i$ for all full D_1 decompositions of irreducibles s takes all generators of A into products of generators of R, and on these elements $t_R=h\circ t_A$. But R is a free abelian group generated by elements which are products of irreducible indecomposables. Hence h extends uniquely to a group homomorphism. In addition, $\mathbf{D}_1$ is coadditive (factorable) iff every group generator of R is the (unique) codomain of some morphism $s<\bigtimes s_j$ iff h' is surjective (bijective) between (group) generators of A and R.

PROPOSITION 3.6. *Let* $S'\subset S$ *be hereditary for* $\mathbf{D}(S)$. *Then if* $\mathbf{D}(S)$ *is unique so is the subdecomposition* $\mathbf{D}(S')$ *and the associated Tutte-Grothendieck group* A' *is the free abelian group whose set of generators is the set of those generators for A in* S'. *Further the natural function* $f':S'\to A'$ *is the restriction of* $t:S\to A$ *to* S' *and* A'.

Similar results hold if $\mathbf{B}(S')$ *is a subbidecomposition of a distributive bidecomposition.*

Proof. From the definition of subdecomposition in (2.8) it follows that refinability, indecomposability, irreducibility, finiteness, uniqueness, and distributivity are all preserved by $\mathbf{D}(S')$ (or $\mathbf{B}(S')$). The rest of the proposition follows because $t:S\to A$ when restricted to S' is universally $\mathbf{D}(S')$-invariant and may be considered to take its values in A'.

PROPOSITION 3.7. *Let* $\mathbf{D}(S)$ *be a unique decomposition. An operator F on S (function from S into itself) is a functor from the category* $\mathbf{D}(S)$ *into* $\mathbf{D}(S)$ *(i.e., is order preserving) if and only if there is an endomorphism h on the Tutte-Grothendieck group that makes the following diagram commute:*



Similar results hold for distributive bidecompositions.

Proof. F is a functor iff for every decomposition $s<\sum_i s_i$, $F(s)<\sum_i F(s_i)\in$ **Diff** $f=t\circ F$ is a $\mathbf{D}$-invariant into the Tutte-Grothendieck group iff there is a group homomorphism h such that $f=h\circ t$.

4. Examples of decomposition categories

Unique decompositions and distributive bidecompositions abound in all branches of mathematics. In this section we list a few examples and sketch the details of the associated Tutte-Grothendieck constructions.

EXAMPLE 4.1. Let $V_1, V_2, \ldots$ denote the set of all (inequivalent) finite dimensional vector spaces over a field k, and let S_i be the set of all linear operators on V_i. On the set $S = \bigcup_i S_i$ we define the decomposition $\mathbf{D}(S)$ generated by the morphisms $T < T'_W + T''_W$ for all linear operators T and proper T-invariant subspaces W where T'_W is the restriction of T to W and T''_W is the induced operator on V/W: $T''(v+W) = T(v) + W$. This decomposition is strictly finite (consider the dimension of V) and refinable (since if W_1 and W_2 are T-invariant subspaces then the restriction of T to $W_1 \cap W_2$, $W_1/W_1 \cap W_2$, $W_2/W_1 \cap W_2$, and $V/W_1 + W_2$ are all well-defined and commonly refine both $T'_{W_1} + T''_{W_1}$ and $T'_{W_2} + T''_{W_2}$).

Assume the underlying field k is algebraically closed. Then the irreducible operators are those defined on a one-dimensional space and must be linear multiples of the identity, $T = \alpha I$. If $T < \sum_{i=1}^{n} T_i$ is a full decomposition of an operator T defined on a vector space V where $T_i = \alpha_i I$ then the dimension of V is an integer valued $\mathbf{D}$-invariant ($h(T_i) = 1$), and the trace of T is an invariant evaluated in k viewed as an additive group such that on irreducibles, $Tr(\alpha_i I) = \alpha_i$.

Further, the *characteristic polynomial* $f(M, \lambda)$ is a multiplicative $\mathbf{D}$-invariant from S into the transcendental extension field $k[\lambda]$ (with 0 deleted) corresponding to the evaluation $f(\alpha_i I) = \lambda - \alpha_i$. Invertible linear operators form a subdecomposition $\mathbf{D}(S')$ in which the *determinant* $d(T)$ is a $\mathbf{D}'$-invariant into the multiplicative group $k\backslash\{0\}$ ($d(\alpha_i I) = \alpha_i$).

EXAMPLE 4.2. Let $\mathbf{D}_1(\mathbf{M})$ denote the direct sum decomposition on the set of all finite square matrices over a commutative ring, $M_1 \oplus M_2 < M_1 \times M_2$ and let $\mathbf{D}_2(\mathbf{M})$ be the decomposition generated by $M < (a_{ij} M_{ij}) + (M - a_{ij} E_{ij})$ for all (but one by one) matrices M and for all nonzero entries a_{ij} of M where M_{ij} is the minor of M formed by deleting the ith row and jth column, and where $M - a_{ij} E_{ij}$ is the matrix M with a_{ij} replaced by 0.

Then $\mathbf{D}_1$ distributes over $\mathbf{D}_2$ forming the distributive bidecomposition $\mathbf{B}(\mathbf{M})$. The irreducible indecomposables are the scalar (one by one) matrices λ_i and if $t(M)$ is the image of a matrix M in the Tutte-Grothendieck ring associated with $\mathbf{B}$, then $t(M)$ is the *permanent* of M.

Note that if, instead, for all nonzero a_{ij} we let $\mathbf{D}_2$ be generated by morphisms $M < ((-1)^{i+j} a_{ij} M_{ij}) + (M - a_{ij} E_{ij})$ then $t(M)$ is the *determinant of* M.

If $\mathbf{M}'$ is the set of polynomial multiples of (0, 1)-matrices then the bidecomposition $\mathbf{B}'(\mathbf{M}')$ generated by the matrix direct sum: $p(x)(M_1 \oplus M_2) < p(x) M_1 \times M_2$ and by $\mathbf{D}_2$ morphisms of the form $p(x) M < (x p(x) M_{ij}) + p(x)(M - a_{ij} E_{ij})$ for all $a_{ij} = 1$ is distributive with polynomial multiples of the scalar matrices ($M=0$ or $M=1$) as the irreducible indecomposables.

The *rook polynomial* $R(M) = \sum_k r_k x^k$ is an invariant of $\mathbf{B}'(\mathbf{M}')$ where r_k counts the number of combinations of k unit entries of the (0, 1)-matrix M, no two on a line (row or column) and where $R(0)=1$, $R(1)=1+x$.

EXAMPLE 4.3. In [15] Watts showed that on the set of homeomorphism classes of triangulable spaces with basepoint there is a unique decomposition generated by morphisms of the form $X < A + X/A$ where A is a subspace of X with common basepoint and X/A is the space X with A identified to the basepoint. The associated group is infinite cyclic since the two point discrete space S^0 is the only irreducible.

The *reduced Euler characteristic* μ (the alternating sum of the ranks of the reduced homology groups) is an integer valued invariant of this decomposition taking a value of one on S^0 and so every other integer invariant must be a multiple of μ. It is noted that the decomposition associated with the *smash product* $(X \times Y / X \wedge Y)$ distributes over the bidecomposition although condition (2.6.b) is not satisfied.

EXAMPLE 4.4. In the prototypical work [14] of Tutte a bidecomposition is defined on all isomorphism classes of multigraphs (graphs with multiple edges and loops) in which the unique coadditive decomposition generated by morphisms $M < \bigtimes M_i$ (where $\bigtimes M_i$ is the multiset of connected components of M) distributes over the unique decomposition generated by $M < M_e' + M_e''$ for all multigraphs M and links (edges which are not loops) $e \in M$, in which M_e' is the subgraph of M formed from the deletion of e and M_e'' is the multigraph resulting from the identification of the two vertices incident with e followed by deletion of e.

The irreducible indecomposables x_i are then the multigraphs consisting of i loops incident to a common vertex ($i=0, 1, 2, \ldots$).

The group integer invariant resulting from the evaluation $h(M)=1$ for all $\mathbf{D}_2$-indecomposables M coincides with the number of subtrees (connected subgraphs with no circuits) for connected multigraphs.

Ring invariants include the number of nowhere zero 1-cycles and $(-1)^v \chi(\lambda, M)$ where v is the number of vertices of M, and $\chi(\lambda, M)$ is the *graph chromatic polynomial* of M (which counts the number of proper λ-colorings). The latter invariant corresponds to the evaluation $\chi(x_i) = -\delta(i, 0)\lambda$ on irreducible indecomposables.

EXAMPLE 4.5. If S is the set of all permutation representations of a finite group

G (all G-sets), then we have the decomposition into domains of transitivity: $(G, X) \leqslant \sum_i (G, X_i)$ where each of the X_i's are invariant under the action of G: $GX_i = X_i$. This decomposition is then factorable with the transitive representations (which are all isomorphic to G acting by left multiplication on cosets of a subgroup) as irreducible elements. The cardinality of X and the number of domains of transitivity of (G, X) are two invariants of this decomposition.

A decomposition which distributes over the above decomposition is the cartesian product decomposition $(G, X_1 \times X_2) \leqslant (G, X_1) \times (G, X_2)$ where G acts on $X_1 \times X_2$ componentwise: $g(x_1, x_2) = (gx_1, gx_2)$. It is clear that if x_1 is an element of X_1, then the subset $x_1 \times X_2 = \{(x_1, x_2) \mid x_2 \in X_2\}$ is a block of $(G, X_1 \times X_2)$ (i.e., domain of imprimitivity B such that for all $g \in G$, gB either equals B or is disjoint from it).

In addition, $X_1 \times x_2$ is a block; so that for a representation to be decomposable there must be two blocks B_1 and B_2 such that $|B_1 \cap B_2| = 1$ and $|B_1|\,|B_2| = |X|$. This condition turns out to be sufficient also. For regular representations (G, G) with group action by left multiplication this becomes a condition on subgroups of G: G_1 and G_2 such that $G_1 G_2 = G$ and $G_1 \cap G_2 = 1$. We have seen earlier (2.3) that this decomposition is not refinable although it is certainly finite. If G is any abelian group then the irreducible indecomposable representations are those isomorphic to the representations $(G, G/H)$ where G/H is a cyclic quotient group of prime power order.

EXAMPLE 4.6. If S is the set of isomorphism classes of combinatorial pregeometries (matroids) [5] then direct sum decomposition $\mathbf{D}_1$ forms a distributive bidecomposition with the decomposition $\mathbf{D}_2$: $G < G \backslash e + G/e$ for all points e which are not direct sum factors of G, where $G \backslash e$ is the subgeometry formed by deleting e and G/e is the contraction of G by e. The irreducible indecomposables are then the loop (one point circuit) and isthmus (one point geometry). The universal invariant turns out to coincide with the classical Tutte polynomial [4] and invariants include the characteristic polynomial and Möbius function as well as the number of subsets, spanning sets, bases, independent sets, and nowhere zero cycles [3].

Similar categories which admit useful distributive bidecompositions include base-pointed pregeometries [1] and geometric lattices [2].

EXAMPLE 4.7. In the classical categorical case, S is the set of equivalence classes of objects of an abelian category and $\mathbf{D}_2$ is decomposition by short exact sequences. Finiteness would be a condition (finite length) which would imply that the objects would be artinian and noetherian. Refinability would be describable in terms of completing the following diagram into a commutative grid of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \bar{A}' & & & & \\
 & & & & \downarrow & & & & \\
0 & \rightarrow & A' & \rightarrow & A & \rightarrow & A'' & \rightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \bar{A} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Invariants would include the length of A and its Euler characteristic. Irreducible in this category means simple.

If $\mathbf{D}_1$ is another (coadditive) decomposition $A_1 \oplus A_2 < A_1 + A_2$ then distributivity means that $A_1 \oplus .$ is an exact functor, while (2.6.b) is the somewhat unusual condition that $A_1 \oplus A_2$ is simple if A_1 and A_2 are.

REFERENCES

[1] T. Brylawski, *A combinatorial model for series-parallel networks*, Trans Am. Math. Soc. *154* (1971), 1–22.
[2] T. Brylairski, *The Mobius function as decompostion invariant*, Proc., Waterloo Conference on Mobius Algebras (1971), 143–148.
[3] T. Brylawski, *A decomposition for combinatorial geometries*, Trans. Am. Math. Soc. *171* (1972), 235–282.
[4] H. Crapo, *The Tutte polynomial*, Aequationes Math., *3* (1970), 211–229.
[5] H. Crapo, and G.–C. Rota, *Combinatorial Geometries*, M.I.T. Press (preliminary edition, 1970).
[6] R. G. Dilworth, *A decomposition theorem for partially ordered sets*, Annals of Math *51* (1950) 161–166.
[7] C. Greene, *Weight enumeration and the geometry of linear codes* (to appear).
[8] M. Henley, *Dissection of generating functions*, (to appear).
[9] S. Lang, *Algebra*, Addison-Wesley 1965.
[10] G.–C. Rota, *Hedrick Lectures*, Math. Assoc. of Amer., Ann. Meeting, Toronto, 1967.
[11] W. R. Scott, *Group Theory*, Prentice Hall, 1964.
[12] R. Stanley, *A Brylawski decomposition for finite ordered sets* (to appear).
[13] R. G. Swan, *Algebraic K-Theory*, Lecture Notes in Mathematics *76* (1968).
[14] W. T. Tutte, *A ring in graph theory*, Proc. Camb. Phil. Soc., *43* (1947), 26–40.
[15] C. E. Watts, *On the Euler characteristic of polyhedra*, Proc. Am. Math. Soc., *13* (1962) 304–306.

University of North Carolina at Chapel Hill
Chapel Hill, North Carolina
U.S.A.