Question 8

Give a combinatorial proof of the following identity.

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}$$

Proof. During the time of the old republic, the Jedi masters held many seats in the republic's grand senate. Each year, an election is held for the control of the $m + r$ seats. However, under the reknown leadership of Jedi Master Yoda, all m jedi masters always won m seats of the senate. As only jedi masters are allowed to take a seat in the old republic, the remaining r seats always fell into the hands of corrupt nobles. Each year, m jedi and n nobles competed for $m + r$ seats but without fail, the m jedi all secured seats while the nobles squabbled over the remaining r. The distribution of power seemed absolute until the invasion of Naboo. Following the invasion, the jedi council speculated on the results of the next election. They enumerated the possibilities. There were $\binom{m+n}{m+r}$ ways to distribute the seats. However, being especially cautious, they considered all disjoint possibilities of distributing the council seats as follows:

1. $\binom{m}{0}\binom{n}{r+0}$: The best case, the Jedi maintain their political position.
2. $\binom{m}{1}\binom{n}{r+1}$: The Jedi lose a seat in the senate. Consequently, the nobles gain a seat.
3. $\binom{m}{2}\binom{n}{r+2}$: The Jedi lose two seats and the nobles capture them.
4. ...
5. $\binom{m}{m}\binom{n}{r+m}$: The worst case scenario, the Jedi lose all the seats in the senate and it is now completely controlled by the nobles.

Since computing the arrangements of seats by $\binom{m+n}{m+r}$ is the same as going through all the possible ways the Jedi could lose $k \in \{0, 1, 2, \ldots, m\}$ seats. It follows that

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}.$$

Note. I used the fact that $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$ if the A's are pairwise disjoint. □