Factorization of the Coefficient of x^{13}

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```
var('x,a1,a2,a3,a4,b1,b2,b3,b4,p1,p2,p3,p4,q1,q2,q3,q4')
w0 = SR.wild(0)
sage_server.MAX_STDOUT_SIZE=250000
sage_server.MAX_OUTPUT=250000
#This defines our variables, sets up our wild card and increases max output for\
    the program.
```

(x, a1, a2, a3, a4, b1, b2, b3, b4, p1, p2, p3, p4, q1, q2, q3, q4)

```
E=(x-a1*a2*a3*a4)*(x-a1*a2*a3*b4)*(x-a1*a2*b3*a4)*(x-a1*a2*b3*b4)*(x-a1*b2*a3*\
    a4)*(x-a1*b2*a3*b4)*(x-a1*b2*b3*a4)*(x-a1*b2*b3*b4)*(x-b1*a2*a3*a4)*(x-b1*a2\
    *a3*b4)*(x-b1*a2*b3*a4)*(x-b1*a2*b3*b4)*(x-b1*b2*a3*a4)*(x-b1*b2*a3*b4)*(x-\
    b1*b2*b3*a4)*(x-b1*b2*b3*b4)
F=E.expand()
#Multiplication of the characteristic function, this is not show do to length.
```

```
g13=-F.coefficient(x,13)
g13
#Gathering of the coefficient for x^13 from our expansion of the characteristic\
    function. We take the negative because we handle the sign in the final form\
    of the characteristic.
```

a1^2*a2^2*a3^3*a4^3*b1*b2 + a1*a2^2*a3^3*a4^3*b1^2*b2 + a1^2*a2*a3^3*a4^3*b1*b2^2 +
a1*a2*a3^3*a4^3*b1^2*b2^2 + a1^2*a2^3*a3^2*a4^3*b1*b3 + a1*a2^3*a3^2*a4^3*b1^2*b3 +
a1^3*a2^2*a3^2*a4^3*b2*b3 + 4*a1^2*a2^2*a3^2*a4^3*b1*b2*b3 +
4*a1*a2^2*a3^2*a4^3*b1^2*b2*b3 + a2^2*a3^2*a4^3*b1^3*b2*b3 + a1^3*a2*a3^2*a4^3*b2^2*b3 +
4*a1^2*a2*a3^2*a4^3*b1*b2^2*b3 + 4*a1*a2*a3^2*a4^3*b1^2*b2^2*b3 +
a2*a3^2*a4^3*b1^3*b2^2*b3 + a1^2*a3^2*a4^3*b1*b2^3*b3 + a1*a3^2*a4^3*b1^2*b2^3*b3 +
a1^2*a2^3*a3*a4^3*b1*b3^2 + a1*a2^3*a3*a4^3*b1^2*b3^2 + a1^3*a2^2*a3*a4^3*b2*b3^2 +
4*a1^2*a2^2*a3*a4^3*b1*b2*b3^2 + 4*a1*a2^2*a3*a4^3*b1^2*b2*b3^2 +
a2^2*a3*a4^3*b1^3*b2*b3^2 + a1^3*a2*a3*a4^3*b2^2*b3^2 + 4*a1^2*a2*a3*a4^3*b1*b2^2*b3^2 +
4*a1*a2*a3*a4^3*b1^2*b2^2*b3^2 + a2*a3*a4^3*b1^3*b2^2*b3^2 + a1^2*a3*a4^3*b1*b2^3*b3^2 +
a1*a3*a4^3*b1^2*b2^3*b3^2 + a1^2*a2^2*a4^3*b1*b2*b3^3 + a1*a2^2*a4^3*b1^2*b2*b3^3 +
a1^2*a2*a4^3*b1*b2^2*b3^3 + a1*a2*a4^3*b1^2*b2^2*b3^3 + a1^2*a2^3*a3^3*a4^2*b1*b4 +
a1*a2^3*a3^3*a4^2*b1^2*b4 + a1^3*a2^2*a3^3*a4^2*b2*b4 + 4*a1^2*a2^2*a3^3*a4^2*b1*b2*b4 +
4*a1*a2^2*a3^3*a4^2*b1^2*b2*b4 + a2^2*a3^3*a4^2*b1^3*b2*b4 + a1^3*a2*a3^3*a4^2*b2^2*b4 +
4*a1^2*a2*a3^3*a4^2*b1*b2^2*b4 + 4*a1*a2*a3^3*a4^2*b1^2*b2^2*b4 +
a2*a3^3*a4^2*b1^3*b2^2*b4 + a1^2*a3^3*a4^2*b1*b2^3*b4 + a1*a3^3*a4^2*b1^2*b2^3*b4 +
a1^3*a2^3*a3^2*a4^2*b3*b4 + 4*a1^2*a2^3*a3^2*a4^2*b1*b3*b4 +
4*a1*a2^3*a3^2*a4^2*b1^2*b3*b4 + a2^3*a3^2*a4^2*b1^3*b3*b4 +
4*a1^3*a2^2*a3^2*a4^2*b2*b3*b4 + 13*a1^2*a2^2*a3^2*a4^2*b1*b2*b3*b4 +
13*a1*a2^2*a3^2*a4^2*b1^2*b2*b3*b4 + 4*a2^2*a3^2*a4^2*b1^3*b2*b3*b4 +
4*a1^3*a2*a3^2*a4^2*b2^2*b3*b4 + 13*a1^2*a2*a3^2*a4^2*b1*b2^2*b3*b4 +
13*a1*a2*a3^2*a4^2*b1^2*b2^2*b3*b4 + 4*a2*a3^2*a4^2*b1^3*b2^2*b3*b4 +
a1^3*a3^2*a4^2*b2^3*b3*b4 + 4*a1^2*a3^2*a4^2*b1*b2^3*b3*b4 +
4*a1*a3^2*a4^2*b1^2*b2^3*b3*b4 + a3^2*a4^2*b1^3*b2^3*b3*b4 + a1^3*a2^3*a3*a4^2*b3^2*b4 +
4*a1^2*a2^3*a3*a4^2*b1*b3^2*b4 + 4*a1*a2^3*a3*a4^2*b1^2*b3^2*b4 +
a2^3*a3*a4^2*b1^3*b3^2*b4 + 4*a1^3*a2^2*a3*a4^2*b2*b3^2*b4 +
13*a1^2*a2^2*a3*a4^2*b1*b2*b3^2*b4 + 13*a1*a2^2*a3*a4^2*b1^2*b2*b3^2*b4 +
4*a2^2*a3*a4^2*b1^3*b2*b3^2*b4 + 4*a1^3*a2*a3*a4^2*b2^2*b3^2*b4 +

13*a1^2*a2*a3*a4^2*b1*b2^2*b3^2*b4 + 13*a1*a2*a3*a4^2*b1^2*b2^2*b3^2*b4 +
4*a2*a3*a4^2*b1^3*b2^2*b3^2*b4 + a1^3*a3*a4^2*b2^3*b3^2*b4 +
4*a1^2*a3*a4^2*b1*b2^3*b3^2*b4 + 4*a1*a3*a4^2*b1^2*b2^3*b3^2*b4 +
a3*a4^2*b1^3*b2^3*b3^2*b4 + a1^2*a2^3*a4^2*b1*b3^3*b4 + a1*a2^3*a4^2*b1^2*b3^3*b4 +
a1^3*a2^2*a4^2*b2*b3^3*b4 + 4*a1^2*a2^2*a4^2*b1*b2*b3^3*b4 +
4*a1*a2^2*a4^2*b1^2*b2*b3^3*b4 + a2^2*a4^2*b1^3*b2*b3^3*b4 + a1^3*a2*a4^2*b2^2*b3^3*b4 +
4*a1^2*a2*a4^2*b1*b2^2*b3^3*b4 + 4*a1*a2*a4^2*b1^2*b2^2*b3^3*b4 +
a2*a4^2*b1^3*b2^2*b3^3*b4 + a1^2*a4^2*b1*b2^3*b3^3*b4 + a1*a4^2*b1^2*b2^3*b3^3*b4 +
a1^2*a2^3*a3^3*a4*b1*b4^2 + a1*a2^3*a3^3*a4*b1^2*b4^2 + a1^3*a2^2*a3^3*a4*b2*b4^2 +
4*a1^2*a2^2*a3^3*a4*b1*b2*b4^2 + 4*a1*a2^2*a3^3*a4*b1^2*b2*b4^2 +
a2^2*a3^3*a4*b1^3*b2*b4^2 + a1^3*a2*a3^3*a4*b2^2*b4^2 + 4*a1^2*a2*a3^3*a4*b1*b2^2*b4^2 +
4*a1*a2*a3^3*a4*b1^2*b2^2*b4^2 + a2*a3^3*a4*b1^3*b2^2*b4^2 + a1^2*a3^3*a4*b1*b2^3*b4^2 +
a1*a3^3*a4*b1^2*b2^3*b4^2 + a1^3*a2^3*a3^2*a4*b3*b4^2 + 4*a1^2*a2^3*a3^2*a4*b1*b3*b4^2 +
4*a1*a2^3*a3^2*a4*b1^2*b3*b4^2 + a2^3*a3^2*a4*b1^3*b3*b4^2 +
4*a1^3*a2^2*a3^2*a4*b2*b3*b4^2 + 13*a1^2*a2^2*a3^2*a4*b1*b2*b3*b4^2 +
13*a1*a2^2*a3^2*a4*b1^2*b2*b3*b4^2 + 4*a2^2*a3^2*a4*b1^3*b2*b3*b4^2 +
4*a1^3*a2*a3^2*a4*b2^2*b3*b4^2 + 13*a1^2*a2*a3^2*a4*b1*b2^2*b3*b4^2 +
13*a1*a2*a3^2*a4*b1^2*b2^2*b3*b4^2 + 4*a2*a3^2*a4*b1^3*b2^2*b3*b4^2 +
a1^3*a3^2*a4*b2^3*b3*b4^2 + 4*a1^2*a3^2*a4*b1*b2^3*b3*b4^2 +
4*a1*a3^2*a4*b1^2*b2^3*b3*b4^2 + a3^2*a4*b1^3*b2^3*b3*b4^2 + a1^3*a2^3*a3*a4*b3^2*b4^2 +
4*a1^2*a2^3*a3*a4*b1*b3^2*b4^2 + 4*a1*a2^3*a3*a4*b1^2*b3^2*b4^2 +
a2^3*a3*a4*b1^3*b3^2*b4^2 + 4*a1^3*a2^2*a3*a4*b2*b3^2*b4^2 +
13*a1^2*a2^2*a3*a4*b1*b2*b3^2*b4^2 + 13*a1*a2^2*a3*a4*b1^2*b2*b3^2*b4^2 +
4*a2^2*a3*a4*b1^3*b2*b3^2*b4^2 + 4*a1^3*a2*a3*a4*b2^2*b3^2*b4^2 +
13*a1^2*a2*a3*a4*b1*b2^2*b3^2*b4^2 + 13*a1*a2*a3*a4*b1^2*b2^2*b3^2*b4^2 +
4*a2*a3*a4*b1^3*b2^2*b3^2*b4^2 + a1^3*a3*a4*b2^3*b3^2*b4^2 +
4*a1^2*a3*a4*b1*b2^3*b3^2*b4^2 + 4*a1*a3*a4*b1^2*b2^3*b3^2*b4^2 +
a3*a4*b1^3*b2^3*b3^2*b4^2 + a1^2*a2^3*a4*b1*b3^3*b4^2 + a1*a2^3*a4*b1^2*b3^3*b4^2 +
a1^3*a2^2*a4*b2*b3^3*b4^2 + 4*a1^2*a2^2*a4*b1*b2*b3^3*b4^2 +
4*a1*a2^2*a4*b1^2*b2*b3^3*b4^2 + a2^2*a4*b1^3*b2*b3^3*b4^2 + a1^3*a2*a4*b2^2*b3^3*b4^2 +
4*a1^2*a2*a4*b1*b2^2*b3^3*b4^2 + 4*a1*a2*a4*b1^2*b2^2*b3^3*b4^2 +
a2*a4*b1^3*b2^2*b3^3*b4^2 + a1^2*a4*b1*b2^3*b3^3*b4^2 + a1*a4*b1^2*b2^3*b3^3*b4^2 +
a1^2*a2^2*a3^3*b1*b2*b4^3 + a1*a2^2*a3^3*b1^2*b2*b4^3 + a1^2*a2*a3^3*b1*b2^2*b4^3 +
a1*a2*a3^3*b1^2*b2^2*b4^3 + a1^2*a2^3*a3^2*b1*b3*b4^3 + a1*a2^3*a3^2*b1^2*b3*b4^3 +
a1^3*a2^2*a3^2*b2*b3*b4^3 + 4*a1^2*a2^2*a3^2*b1*b2*b3*b4^3 +
4*a1*a2^2*a3^2*b1^2*b2*b3*b4^3 + a2^2*a3^2*b1^3*b2*b3*b4^3 + a1^3*a2*a3^2*b2^2*b3*b4^3 +
4*a1^2*a2*a3^2*b1*b2^2*b3*b4^3 + 4*a1*a2*a3^2*b1^2*b2^2*b3*b4^3 +
a2*a3^2*b1^3*b2^2*b3*b4^3 + a1^2*a3^2*b1*b2^3*b3*b4^3 + a1*a3^2*b1^2*b2^3*b3*b4^3 +
a1^2*a2^3*a3*b1*b3^2*b4^3 + a1*a2^3*a3*b1^2*b3^2*b4^3 + a1^3*a2^2*a3*b2*b3^2*b4^3 +
4*a1^2*a2^2*a3*b1*b2*b3^2*b4^3 + 4*a1*a2^2*a3*b1^2*b2*b3^2*b4^3 +
a2^2*a3*b1^3*b2*b3^2*b4^3 + a1^3*a2*a3*b2^2*b3^2*b4^3 + 4*a1^2*a2*a3*b1*b2^2*b3^2*b4^3 +
4*a1*a2*a3*b1^2*b2^2*b3^2*b4^3 + a2*a3*b1^3*b2^2*b3^2*b4^3 + a1^2*a3*b1*b2^3*b3^2*b4^3 +
a1*a3*b1^2*b2^3*b3^2*b4^3 + a1^2*a2^2*b1*b2*b3^3*b4^3 + a1*a2^2*b1^2*b2*b3^3*b4^3 +
a1^2*a2*b1*b2^2*b3^3*b4^3 + a1*a2*b1^2*b2^2*b3^3*b4^3

```
g13.factor()
#This is an initial factoring to try and make things easier.
```

(a1*a2*a3^2*a4^2*b1*b2 + a1*a2^2*a3*a4^2*b1*b3 + a1^2*a2*a3*a4^2*b2*b3 +
a1*a2*a3*a4^2*b1*b2*b3 + a2*a3*a4^2*b1^2*b2*b3 + a1*a3*a4^2*b1*b2^2*b3 +
a1*a2*a4^2*b1*b2*b3^2 + a1*a2^2*a3^2*a4*b1*b4 + a1^2*a2*a3^2*a4*b2*b4 +
a1*a2*a3^2*a4*b1*b2*b4 + a2*a3^2*a4*b1^2*b2*b4 + a1*a3^2*a4*b1*b2^2*b4 +
a1^2*a2^2*a3*a4*b3*b4 + a1*a2^2*a3*a4*b1*b3*b4 + a2^2*a3*a4*b1^2*b3*b4 +
a1^2*a2*a3*a4*b2*b3*b4 + 3*a1*a2*a3*a4*b1*b2*b3*b4 + a2*a3*a4*b1^2*b2*b3*b4 +
a1^2*a3*a4*b2^2*b3*b4 + a1*a3*a4*b1*b2^2*b3*b4 + a3*a4*b1^2*b2^2*b3*b4 +
a1*a2^2*a4*b1*b3^2*b4 + a1^2*a2*a4*b2*b3^2*b4 + a1*a2*a4*b1*b2*b3^2*b4 +
a2*a4*b1^2*b2*b3^2*b4 + a1*a4*b1*b2^2*b3^2*b4 + a1*a2*a3^2*b1*b2*b4^2 +
a1*a2^2*a3*b1*b3*b4^2 + a1^2*a2*a3*b2*b3*b4^2 + a1*a2*a3*b1*b2*b3*b4^2 +
a2*a3*b1^2*b2*b3*b4^2 + a1*a3*b1*b2^2*b3*b4^2 + a1*a2*b1*b2*b3^2*b4^2)*(a1 + b1)*(a2 +
b2)*(a3 + b3)*(a4 + b4)

```
t=(a1 + b1)*(a2 + b2)*(a3 + b3)*(a4 + b4)
t
f13=g13.factor()/t
f13
#This seperates the factorization above into two parts.
```
```
(a1 + b1)*(a2 + b2)*(a3 + b3)*(a4 + b4)
a1*a2*a3^2*a4^2*b1*b2 + a1*a2^2*a3*a4^2*b1*b3 + a1^2*a2*a3*a4^2*b2*b3 +
a1*a2*a3*a4^2*b1*b2*b3 + a2*a3*a4^2*b1^2*b2*b3 + a1*a3*a4^2*b1*b2^2*b3 +
a1*a2*a4^2*b1*b2*b3^2 + a1*a2^2*a3^2*a4*b1*b4 + a1^2*a2*a3^2*a4*b2*b4 +
a1*a2*a3^2*a4*b1*b2*b4 + a2*a3^2*a4*b1^2*b2*b4 + a1*a3^2*a4*b1*b2^2*b4 +
a1^2*a2^2*a3*a4*b3*b4 + a1*a2^2*a3*a4*b1*b3*b4 + a2^2*a3*a4*b1^2*b3*b4 +
a1^2*a2*a3*a4*b2*b3*b4 + 3*a1*a2*a3*a4*b1*b2*b3*b4 + a2*a3*a4*b1^2*b2*b3*b4 +
a1^2*a3*a4*b2^2*b3*b4 + a1*a3*a4*b1*b2^2*b3*b4 + a3*a4*b1^2*b2^2*b3*b4 +
a1*a2^2*a4*b1*b3^2*b4 + a1^2*a2*a4*b2*b3^2*b4 + a1*a2*a4*b1*b2*b3^2*b4 +
a2*a4*b1^2*b2*b3^2*b4 + a1*a4*b1*b2^2*b3^2*b4 + a1*a2*a3^2*b1*b2*b4^2 +
a1*a2^2*a3*b1*b3*b4^2 + a1^2*a2*a3*b2*b3*b4^2 + a1*a2*a3*b1*b2*b3*b4^2 +
a2*a3*b1^2*b2*b3*b4^2 + a1*a3*b1*b2^2*b3*b4^2 + a1*a2*b1*b2*b3^2*b4^2
```

```
s=p1*p2*p3*p4
s
#This is our substitution for t.
```
```
p1*p2*p3*p4
```

```
c3=sum(f13.find(3*w0))
c3
s3=3*q1*q2*q3*q4
s3
#This is the term with a coefficient of 3 from f13 and its substitution.
```
```
3*a1*a2*a3*a4*b1*b2*b3*b4
3*q1*q2*q3*q4
```

```
c1=sum(f13.find(1*w0))-c3
c1
#This is the sum of the terms with a coefficient of 1 from f13. We group these\
    by if they have exactly three of four a variables to the first power in \
   them and if they have exactly two of the four a variables to the first power\
    in them.
```
```
a1*a2*a3^2*a4^2*b1*b2 + a1*a2^2*a3*a4^2*b1*b3 + a1^2*a2*a3*a4^2*b2*b3 +
a1*a2*a3*a4^2*b1*b2*b3 + a2*a3*a4^2*b1^2*b2*b3 + a1*a3*a4^2*b1*b2^2*b3 +
a1*a2*a4^2*b1*b2*b3^2 + a1*a2^2*a3^2*a4*b1*b4 + a1^2*a2*a3^2*a4*b2*b4 +
a1*a2*a3^2*a4*b1*b2*b4 + a2*a3^2*a4*b1^2*b2*b4 + a1*a3^2*a4*b1*b2^2*b4 +
a1^2*a2^2*a3*a4*b3*b4 + a1*a2^2*a3*a4*b1*b3*b4 + a2^2*a3*a4*b1^2*b3*b4 +
a1^2*a2*a3*a4*b2*b3*b4 + a2*a3*a4*b1^2*b2*b3*b4 + a1^2*a3*a4*b2^2*b3*b4 +
a1*a3*a4*b1*b2^2*b3*b4 + a3*a4*b1^2*b2^2*b3*b4 + a1*a2^2*a4*b1*b3^2*b4 +
a1^2*a2*a4*b2*b3^2*b4 + a1*a2*a4*b1*b2*b3^2*b4 + a2*a4*b1^2*b2*b3^2*b4 +
a1*a4*b1*b2^2*b3^2*b4 + a1*a2*a3^2*b1*b2*b4^2 + a1*a2^2*a3*b1*b3*b4^2 +
a1^2*a2*a3*b2*b3*b4^2 + a1*a2*a3*b1*b2*b3*b4^2 + a2*a3*b1^2*b2*b3*b4^2 +
a1*a3*b1*b2^2*b3*b4^2 + a1*a2*b1*b2*b3^2*b4^2
```

```
c1123=sum(c1.find(a1*a2*a3*w0))
c1123
c1123.factor()
s1123=-(p4^2+2*q4)*q1*q2*q3
s1123
#This is the group with a1, a2, and a3 all to the first power, its \
   factorization, and its substitution.
```
```
a1*a2*a3*a4^2*b1*b2*b3 + a1*a2*a3*b1*b2*b3*b4^2
(a4^2 + b4^2)*a1*a2*a3*b1*b2*b3
-(p4^2 + 2*q4)*q1*q2*q3
```

```
c1124=sum(c1.find(a1*a2*a4*w0))
```

```
c1124
c1124.factor()
s1124=-(p3^2+2*q3)*q1*q2*q4
s1124
#This is the group with a1, a2, and a4 all to the first power, its \
    factorization, and its substitution.
```
```
a1*a2*a3^2*a4*b1*b2*b4 + a1*a2*a4*b1*b2*b3^2*b4
(a3^2 + b3^2)*a1*a2*a4*b1*b2*b4
-(p3^2 + 2*q3)*q1*q2*q4
```

```
c1134=sum(c1.find(a1*a3*a4*w0))
c1134
c1134.factor()
s1134=-(p2^2+2*q2)*q1*q3*q4
s1134
#This is the group with a1, a3, and a4 all to the first power, its \
    factorization, and its substitution.
```
```
a1*a2^2*a3*a4*b1*b3*b4 + a1*a3*a4*b1*b2^2*b3*b4
(a2^2 + b2^2)*a1*a3*a4*b1*b3*b4
-(p2^2 + 2*q2)*q1*q3*q4
```

```
c1234=sum(c1.find(a2*a3*a4*w0))
c1234
c1234.factor()
s1234=-(p1^2+2*q1)*q2*q3*q4
s1234
#This is the group with a2, a3, and a4 all to the first power, its \
    factorization, and its substitution.
```
```
a1^2*a2*a3*a4*b2*b3*b4 + a2*a3*a4*b1^2*b2*b3*b4
(a1^2 + b1^2)*a2*a3*a4*b2*b3*b4
-(p1^2 + 2*q1)*q2*q3*q4
```

```
c112=sum(c1.find(a1*a2*w0))-c1123-c1124
c112
c112.factor()
s112=(p3^2 + 2*q3)*(p4^2+2*q4)*q1*q2
s112
#This is the group with only a1 and a2 to the first power, its factorization, \
    and its substitution.
```
```
a1*a2*a3^2*a4^2*b1*b2 + a1*a2*a4^2*b1*b2*b3^2 + a1*a2*a3^2*b1*b2*b4^2 +
a1*a2*b1*b2*b3^2*b4^2
(a3^2 + b3^2)*(a4^2 + b4^2)*a1*a2*b1*b2
(p3^2 + 2*q3)*(p4^2 + 2*q4)*q1*q2
```

```
c113=sum(c1.find(a1*a3*w0))-c1123-c1134
c113
c113.factor()
s113=(p2^2 + 2*q2)*(p4^2+2*q4)*q1*q3
s113
#This is the group with only a1 and a3 to the first power, its factorization, \
    and its substitution.
```
```
a1*a2^2*a3*a4^2*b1*b3 + a1*a3*a4^2*b1*b2^2*b3 + a1*a2^2*a3*b1*b3*b4^2 +
a1*a3*b1*b2^2*b3*b4^2
(a2^2 + b2^2)*(a4^2 + b4^2)*a1*a3*b1*b3
(p2^2 + 2*q2)*(p4^2 + 2*q4)*q1*q3
```

```
c114=sum(c1.find(a1*a4*w0))-c1124-c1134
c114
c114.factor()
s114=(p2^2 + 2*q2)*(p3^2+2*q3)*q1*q4
s114
```

```
#This is the group with only a1 and a4 to the first power, its factorization, \
    and its substitution.
```
```
a1*a2^2*a3^2*a4*b1*b4 + a1*a3^2*a4*b1*b2^2*b4 + a1*a2^2*a4*b1*b3^2*b4 +
a1*a4*b1*b2^2*b3^2*b4
(a2^2 + b2^2)*(a3^2 + b3^2)*a1*a4*b1*b4
(p2^2 + 2*q2)*(p3^2 + 2*q3)*q1*q4
```

```
c123=sum(c1.find(a2*a3*w0))-c1123-c1234
c123
c123.factor()
s123=(p1^2 + 2*q1)*(p4^2+2*q4)*q2*q3
s123
#This is the group with only a2 and a3 to the first power, its factorization, \
    and its substitution.
```
```
a1^2*a2*a3*a4^2*b2*b3 + a2*a3*a4^2*b1^2*b2*b3 + a1^2*a2*a3*b2*b3*b4^2 +
a2*a3*b1^2*b2*b3*b4^2
(a1^2 + b1^2)*(a4^2 + b4^2)*a2*a3*b2*b3
(p1^2 + 2*q1)*(p4^2 + 2*q4)*q2*q3
```

```
c124=sum(c1.find(a2*a4*w0))-c1124-c1234
c124
c124.factor()
s124=(p1^2 + 2*q1)*(p3^2+2*q3)*q2*q4
s124
#This is the group with only a2 and a4 to the first power, its factorization, \
    and its substitution.
```
```
a1^2*a2*a3^2*a4*b2*b4 + a2*a3^2*a4*b1^2*b2*b4 + a1^2*a2*a4*b2*b3^2*b4 +
a2*a4*b1^2*b2*b3^2*b4
(a1^2 + b1^2)*(a3^2 + b3^2)*a2*a4*b2*b4
(p1^2 + 2*q1)*(p3^2 + 2*q3)*q2*q4
```

```
c134=sum(c1.find(a3*a4*w0))-c1134-c1234
c134
c134.factor()
s134=(p1^2 + 2*q1)*(p2^2+2*q2)*q3*q4
s134
#This is the group with only a3 and a4 to the first power, its factorization, \
    and its substitution.
```
```
a1^2*a2^2*a3*a4*b3*b4 + a2^2*a3*a4*b1^2*b3*b4 + a1^2*a3*a4*b2^2*b3*b4 +
a3*a4*b1^2*b2^2*b3*b4
(a1^2 + b1^2)*(a2^2 + b2^2)*a3*a4*b3*b4
(p1^2 + 2*q1)*(p2^2 + 2*q2)*q3*q4
```

```
L=s*(s1123+s1124+s1134+s1234+s112+s113+s114+s123+s124+s134+s3)
L
#This line brings all the group substitutions back into one expression.
```
```
((p3^2 + 2*q3)*(p4^2 + 2*q4)*q1*q2 + (p2^2 + 2*q2)*(p4^2 + 2*q4)*q1*q3 + (p1^2 +
2*q1)*(p4^2 + 2*q4)*q2*q3 - (p4^2 + 2*q4)*q1*q2*q3 + (p2^2 + 2*q2)*(p3^2 + 2*q3)*q1*q4 +
(p1^2 + 2*q1)*(p3^2 + 2*q3)*q2*q4 - (p3^2 + 2*q3)*q1*q2*q4 + (p1^2 + 2*q1)*(p2^2 +
2*q2)*q3*q4 - (p2^2 + 2*q2)*q1*q3*q4 - (p1^2 + 2*q1)*q2*q3*q4 + 3*q1*q2*q3*q4)*p1*p2*p3*p4
```

```
L1=L(p1=a1+b1, p2=a2+b2, p3=a3+b3, p4=a4+b4, q1=-1*a1*b1, q2=-1*a2*b2, q3=-1*a3\
    *b3, q4=-1*a4*b4)
bool(L1==g13)
#This is a check to make sure that when back substitute we end up with the \
    original coefficient.
```
```
True
```

```
L.expand()
latex(L.expand())
```

```
#This is the expansion of our coefficient and a latex output for easy copying \
    to the thesis paper.
```

```
p1*p2*p3^3*p4^3*q1*q2 + p1*p2^3*p3*p4^3*q1*q3 + p1^3*p2*p3*p4^3*q2*q3 +
p1*p2^3*p3^3*p4*q1*q4 + p1^3*p2*p3^3*p4*q2*q4 + p1^3*p2^3*p3*p4*q3*q4 +
5*p1*p2*p3*p4^3*q1*q2*q3 + 5*p1*p2*p3^3*p4*q1*q2*q4 + 5*p1*p2^3*p3*p4*q1*q3*q4 +
5*p1^3*p2*p3*p4*q2*q3*q4 + 19*p1*p2*p3*p4*q1*q2*q3*q4
```

```
p_{1} p_{2} p_{3}^{3} p_{4}^{3} q_{1} q_{2} + p_{1} p_{2}^{3} p_{3} p_{4}^{3} q_{1} q_{3}
+ p_{1}^{3} p_{2} p_{3} p_{4}^{3} q_{2} q_{3} + p_{1} p_{2}^{3} p_{3}^{3} p_{4} q_{1}
q_{4} + p_{1}^{3} p_{2} p_{3}^{3} p_{4} q_{2} q_{4} + p_{1}^{3} p_{2}^{3} p_{3} p_{4}
q_{3} q_{4} + 5 \, p_{1} p_{2} p_{3} p_{4}^{3} q_{1} q_{2} q_{3} + 5 \, p_{1} p_{2}
p_{3}^{3} p_{4} q_{1} q_{2} q_{4} + 5 \, p_{1} p_{2}^{3} p_{3} p_{4} q_{1} q_{3} q_{4} + 5
\, p_{1}^{3} p_{2} p_{3} p_{4} q_{2} q_{3} q_{4} + 19 \, p_{1} p_{2} p_{3} p_{4} q_{1}
q_{2} q_{3} q_{4}
```