Functors in Differential Geometry

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Abstract

After presenting the notion of a category and providing some examples we focus towards the cotangent bundle of a manifold. This space plays a very important role in symplectic geometry. Indeed we will show that it is possible to give a symplectic structure on the cotangent bundle in a canonical way. We present a functor between the category of smooth manifolds equipped with diffeomorphisms (or local diffeomorphisms) and the category of symplectic manifolds equipped with symplectomorphisms for which the object assignment consists of associating to a space its cotangent bundle.

Contents

1 Introduction

Category theory is the study of mathematical structures and relations between them. It provides a common language to almost all areas of mathematics, its power as a language can be compared to the importance of basic set theory, but in a more general framework. Indeed we can for instance consider the category of manifolds while the class of all manifolds is not a set. It is also interesting to look at the relationships between categories which we call *functors*. Moreover it happens quite often in mathematics that one talks about natural constructions or transformations. Category theory allows to define what natural means via the notion of *natural transformations* which can be thought as the relationships between functors of two categories. As a consequence it allows to actually prove that a construction is natural which is very handy for mathematicians. Indeed the word natural comes up quite often in mathematical constructions. A natural construction is usually understood as a new construction that emerges directly (naturally) from previous operations and constructions, and for which we feel like there is just one way to do the actual construction (e.g. if V is a vector space of dimension n there is a natural isomorphism between V and V^{**}). More precisely, in category theory, one talks about *objects* and *morphisms* (Figure 1), which are maps between these objects, both satisfying the defining properties of the category. A well-known example of a category is the category of groups, together with the definition of a group homomorphism. After defining these notions, we present some important and familiar examples of categories and functors in Differential Geometry and Algebraic Topology. The most relevant functor for our project is the cotangent functor. We will see how the contangent bundle can be viewed as a functor between the category of Smooth Manifolds (equipped with diffeomorphisms) and Symplectic Manifolds (equipped with symplectomorphisms) by proving the following theorem (in section 4.2):

Theorem 4.7: *The canonical symplectic form on the cotangent bundle is invariant under diffeomorphisms in the following sense : Suppose Q and $\tilde{Q}$ are smooth manifolds and $F : Q \to \tilde{Q}$ a diffeomorphism. Let $\mathcal{T}^*F : T^*\tilde{Q} \to T^*Q$ be the map described in section 3.3.5. $\mathcal{T}^*F$ is a symplectomorphism when both T^*Q and $T^*\tilde{Q}$ are endowed with their canonical symplectic forms.*

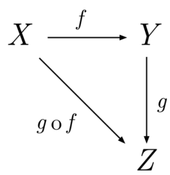

Figure 1: Schematic representation of a category with objects X, Y, Z and morphisms $f, g, g \circ f$. (Identities would appear as three arrows, from the letters X, Y, and Z to themselves)

Y. Eliashberg, a symplectic geometer, asked a natural question following the construction of the cotangent functor, precisely is it injective in the following sense:

Question: *If two manifolds Q and $\tilde{Q}$ are such that T^*Q and $T^*\tilde{Q}$ are symplectomorphic then are Q and $\tilde{Q}$ diffeomorphic?*
This is a hard question that is still an open problem in differential geometry. F. Ziltener and Y. Karshon [4] answer this question in a particular case. Namely, if G is a compact and connected Lie group, they classify the Hamiltonian Lie group actions up to equivariant symplectomorphisms, by providing an injective functor between the category of Hamiltonian Lie group actions and the category of symplectic representation of closed subgroup of G, an Hamiltonian Lie group action being canonically symplectomorphic to the cotangent bundle symplectic quotient, this provides an answer to the question in this precise case. Another case for which injectivity holds, is the case of exotic spheres of odd dimension ≥ 5. The first breakthrough was made by Abouzaid[7] who showed that for $4k+1$ dimensional homotopy spheres that do not bound a parallelisable manifold, the cotangent bundle of these spheres is not diffeomorphic to T^*S^{4k+1}. Later this result was enhanced by Ekholm et al [8] who proved that up to orientation preserving diffeomorphism all exotic spheres have distinct cotangent bundles.

2 A brief history of Vector Bundles and Symplectic Manifolds

The vast development of physics in the early years of the 20th century had a huge impact in mathematics. "The astonishing fact that gravitation is just a manifestation of the curvature of spacetime made a deep impression on mathematicians like Elie Cartan and Hermann Weyl. Especially Weyl, who was a student of Hilbert at that time". [5]
His initial work was on analysis and spectral theory, but he made a lot of contributions to differential geometry too, motivated by the work of the Italian geometers, like Ricci and Levi-Civita, who had studied deeply the aspects of curvature and tensor calclus on Riemannian manifolds. A serious obstruction in doing physics or mathematics on manifolds is that the result have to be checked in all the coordinates systems of the manifold, if the manifold itself do not have an intrinsic defined coordinate system. A method to overcome this obstruction is by making computations with tensors and their derivatives. That coordinate-invariant differential calculus of tensors can be realized via the existence of a *connection* or, equivalently, an isomorphism between the tangent spaces at a neighbourhood of a point. This isomorphism is now called the *Parallel transport*.
Parallel transport is a way of moving tangent vectors along curves without changing their angle and their length. By define parallel transport one can define derivatives of vector and tensor fields. But Levi-Civita, who has already used this isomorhism, only worked for Riemannian manifolds that were embedded in an Euclidean space. It was Weyl who defined the parallel transport on any Riemannian manifolds not necessarily embedded in an Euclidean space and not only that, he also observed that in order to define the parallel transport it was necessary that the manifold should have a metric. "Therefore he defined the concept of parallel transport in a more axiomatic way. He created the notion of an *affine connection* on a Riemannian manifold". [5]
Weyl then tried to generalize the concept of parallel transport by scaling the length of the vectors and by considering the possible scales at the various points as a bundle on the manifold. He introduced connections on this bundle and with the use of differential forms defined on the manifold he expressed the curvature of this bundle. This vector bundle is called the *scale bundle*, as it is defined in [5], and it was one of the first appearances of vector bundles in mathematics, apart from the usual tangent bundle.

"The origin of Symplectic Geometry lies in Classical Mechanics", [6]. The equations of motion in Classical Mechanics arise as solutions of variational problems. A characteristic example is Fermat's principle of least time which states that light moves from one point to another by a path which takes the shortest amount of time. Similarly, all systems that possess kinetic but not potential energy move along geodesics, which are paths minimizing length and energy.
If a system has kinetic and potential energy then there is a quantity which minimize the mean value of kinetic and potential energy. This quantity is called the *action*. The paths that minimize some action functional are solutions of a system of n second order differential equations called the *Euler-Lagrange* equations of the variational problem. The study of Euler-Lagrange equations of one-dimensional variation and the Hamiltonian formalism leads to the notion of a *canonical transformation* which preserves Hamilton's form of the equations of motion. Based on this observation one can define a manifold equipped with an atlas whose transition maps are the canonical transformations. Such a manifold is called *symplectic*. This structure give rise to a differential form which is called the *symplectic form*.
A manifold equipped with a symplectic form is called a *symplectic manifold*. The symplectic form is closed and non-degenerate and from the skew-symmetry of differential forms it follows that manifolds carrying such forms have to be even dimensional. The canonical transformations are called *symplectomorphisms* and they are precisely the maps that preserve the symplectic form of a manifold. "The mathematical formulation of classical mechanics gave rise to this beautiful subject of mathematics that is now called Symplectic Geometry and Topology", [6]. which is the subject of mathematics that studies manifolds equipped with symplectic structure. An astonishing fact is that the general theory of vector bundles had a huge impact on the development of symplectic geometry. More precisely the contagent bundle of every smooth manifold has a natural symplectic structure which is one of the main topics of this report, (section 4.2). Also, symplectic Geometry has many applications apart from physics, also in other areas of mathematics like low-dimensional topology, homological algebra, representation theory, algebraic and complex geometry and category theory. [6]

3 Categories and Functors

3.1 Definitions

In order to read this report a basic understanding of category theory is needed. Here we introduce three important concepts: categories, functors and natural transformations.

Definition 3.1.1. A category $\mathcal{C}$ is a collection of *objects* $\mathbf{Ob}(\mathcal{C})$, a collection of *morphisms* $\mathbf{Hom}(\mathcal{C})$ between those objects and a composition operator $\circ$ defined on the morphisms. We denote the morphisms between objects $X,\ Y \in \mathbf{Ob}(\mathcal{C})$ by $\mathbf{Hom}_{\mathcal{C}}(X,\ Y)$. Furthermore, the following properties hold:

1. Composition of morphisms is associative: For objects $W,\ X,\ Y,\ Z \in \mathbf{Ob}(\mathcal{C})$ and morphisms $f : Y \to Z$, $g : X \to Y$ and $h : W \to X$ (so $f \in \mathbf{Hom}_{\mathcal{C}}(Y,\ Z)$ etc. . .) we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. For every object there exists an identity: For every $X \in \mathbf{Ob}(\mathcal{C})$ there exists a function $1_X \in \mathbf{Hom}_{X,\ X}(\mathcal{C})$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any morphisms $f,\ g$ with codomain X and domain X respectively.

△

The notation $\mathbf{Ob}(\mathcal{C})$ and $\mathbf{Hom}(\mathcal{C})$ will be used in this section in order to define category theoretical concepts formally, but in the remainder of this report the notation will be dropped and the categories with their respective objects, morphisms and compositions will be clear.

Example 3.1.2. A very simple example is the category of sets and their maps: **Sets** where the objects are sets, maps between sets are morphisms and the standard composition is the composition operator. Of course, one can restrict the sets to topological spaces, metric spaces, groups and so forth, while restricting the maps to continuous maps or homomorphisms. In those cases there is still an identity map and composition is still associative. △

Next, we need the concept of a *map between categories.*

Definition 3.1.3. Let $\mathcal{C},\ \mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a map that works on objects and morphisms by $F : \mathbf{Ob}(\mathcal{C}) \to \mathbf{Ob}(\mathcal{D})$ and $F : \mathbf{Hom}(\mathcal{C}) \to \mathbf{Hom}(\mathcal{D})$ such that the category structure is preserved:

1. If $f \in \mathbf{Hom}_{\mathcal{C}}(X,\ Y),\ g \in \mathbf{Hom}_{\mathcal{C}}(Y,\ Z)$ then $F(g \circ f) = F(g) \circ F(f)$. In this case F is called a *covariant* functor.

2. For every $X \in \mathbf{Ob}(\mathcal{C})$ we have $F(1_X) = 1_{F(X)}$.

△

More specifically, the above definition is that of a *covariant* functor. This means that the order of composition is preserved. This is in contrast with a *contravariant* functor for which every morphism $f : X \to Y$ is sent to $F(f) : F(Y) \to F(X)$. As a result $F(g \circ f) = F(f) \circ F(g)$.

Lastly, one can map functors to functors by means of a *natural transformation.*

Definition 3.1.4. Given two functors $F,\ G : \mathcal{C} \to \mathcal{D}$ a *natural transformation* $\eta : F \to G$ is a family of morphisms $\{\eta_X : F(X) \to G(X)\}_{X \in \mathbf{Ob}(\mathcal{C})}$ in $\mathbf{Hom}(\mathcal{D})$ such that the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \big\downarrow{\scriptstyle \eta_X} & & \big\downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

If the functors F, G are contravariant $F(f) : F(Y) \to F(X)$ and $G(f) : G(Y) \to G(X)$ and therefore the horizontal arrows in the above diagram should be flipped.

$\triangle$

Example 3.1.5. The definition of a functor and a natural transformation allow us to consider the collection of functors $\{F : \mathcal{C} \to \mathcal{D}\}$ and their natural transformations as a category. The functors are the objects, and the natural transformations are the morphisms. Such a category is called a *functor category.*

$\triangle$

3.2 Examples in Algebraic Topology

3.2.1 Linearization

Let's consider the two following categories:

- **Set** The objects are sets and where the morphisms are set maps.
- **Ab** The objects are Abelian groups and where the morphisms are group homomorphisms.

Consider a set X and an Abelian group A, then the A-linearization of X, denoted as $A[X]$, is defined as :

$$A[X] = \{f : X \to A | \text{card}(f^{-1}(A - \{0\})) < \infty\}.$$

The A-linearization $A[X]$ inherits the Abelian group structure from A, and it is useful to see elements of $A[X]$ as a formal finite linear combination with coefficients in A. If $f \in A[X]$, it is non-zero on a finite set of points of X, say $\{x_1, \ldots, x_n\}$. Denoting $f(x_1), \ldots, f(x_n)$ by $a_1, \ldots, a_n$ we can represent f as

$$f = \sum_{i=1}^{n} a_i \cdot x_i.$$

From a set map $\gamma : X \to Y$ we get an induced map

$$\gamma_* : \begin{pmatrix} A[X] \to A[Y] \\ f \to \gamma_*(f) \end{pmatrix}$$

where

$$\gamma_*(f) = \sum_{i=1}^{n} a_i \cdot \gamma(x_i).$$

Notice that, for any $f \in A[X]$, we have $(\text{Id}_X)_*(f) = f$. Also it is clear that $(\gamma_1 \circ \gamma_2)_* = (\gamma_1)_* \circ (\gamma_2)_*$. Hence the rule $\mathcal{F}$ defined below is a covariant functor.

$$\mathcal{F}_{ob} : \begin{pmatrix} \textbf{Set} \to \textbf{Ab} \\ X \to A[X] \end{pmatrix}$$

$$\mathcal{F}_{hom} : \begin{pmatrix} \textbf{Set Maps} \to \textbf{Group Homomorphisms} \\ \gamma \to \gamma_* \end{pmatrix}$$

3.2.2 Fundamental Group

Consider the two following categories:

- $\textbf{Top}_*$ where objects are pointed topological spaces and morphisms are pointed continuous maps, and
- **Grp** where objects are groups and morphisms are group homomorphisms.

A continuous map $f : (X, p) \to (Y, f(p))$ between pointed topological spaces induces a group homomorphism $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ defined as:

$$f_* : \begin{pmatrix} \pi_1(X, p) \to \pi_1(Y, f(p)) \\ [\alpha] \to [f\alpha] \end{pmatrix}$$

It's direct to prove that that f_* is a well-defined homomorphism and that $(\mathrm{Id}_{(X,p)})_* = \mathrm{Id}_{\pi_1(X,p)}$ and that $(f \circ g)_* = f_* \circ g_*$. Hence the rule $\mathcal{F}$ defined below is a covariant functor.

$$\mathcal{F}_{ob} : \begin{pmatrix} \mathbf{Top}_* \to \mathbf{Grp} \\ (X,p) \to \pi_1(X,p) \end{pmatrix}$$

$$\mathcal{F}_{hom} : \begin{pmatrix} \textbf{Pointed Continuous Maps} \to \textbf{Group Homomorphisms} \\ (f : (X,p) \to (Y, f(p)) \to f_* : \pi_1(X,p) \to \pi_1(Y, f(p)) \end{pmatrix}$$

3.3 Examples in Differential Geometry

3.3.1 Smooth Maps

Let's consider two new categories:

- $\mathbf{Man}_{\mathrm{Smooth}}$: The category where the objects are smooth manifolds and the morphisms are smooth maps.
- $\mathbf{Vec}_{\mathbb{R}}$: The category where the objects are real vector spaces and the morphisms are linear maps.

For a manifold M, we denote by $\mathcal{C}^\infty(M)$ the set of smooth maps from M to $\mathbb{R}$. The set $\mathcal{C}^\infty(M)$ has a $\mathbb{R}$-vector space structure by pointwise addition and scalar multiplication. Let's now consider two smooth manifolds M, N and a smooth map $f : M \to N$. Then f induces a linear map f_* defined as

$$f_* : \begin{pmatrix} \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M) \\ g \to g \circ f \end{pmatrix}$$

From this definition it is clear that $(\mathrm{Id}_M)_* = \mathrm{Id}_{\mathcal{C}^\infty(M)}$ and also that $(f \circ g)_* = g_* \circ f_*$. This defines the rule $\mathcal{F}$ below, which is a contravariant functor:

$$\mathcal{F}_{ob} : \begin{pmatrix} \mathbf{Man}_{\mathrm{Smooth}} \to \mathbf{Vec}_{\mathbb{R}} \\ M \to \mathcal{C}^\infty(M) \end{pmatrix}$$

$$\mathcal{F}_{hom} : \begin{pmatrix} \textbf{Smooth Maps} \to \textbf{Linear Maps} \\ f \to f^* \end{pmatrix}$$

3.3.2 Tangent Spaces

In this example we consider the category $\mathbf{Vec}_{\mathbb{R}}$ and a new one:

- $\mathbf{Man}_{\mathrm{Smooth}*}$: The objects are pointed smooth manifolds and the morphisms are pointed smooth maps.

A smooth map $f : (M,p) \to (N, f(p))$ induces a linear map called the differential of f at p, denoted by $df_p : T_pM \to T_{f(p)}N$. We have that $d(Id_M)_p = \mathrm{Id}_{T_pM}$. Furthermore the composition rule ensures that $d(g \circ f)_p = dg_{f(p)} \circ df_p$. Hence the rule $\mathcal{F}$ defined below is a covariant functor.

$$\mathcal{F}_{ob} : \begin{pmatrix} \mathbf{Man}_{\mathrm{Smooth}*} \to \mathbf{Vec}_{\mathbb{R}} \\ (M,p) \to T_pM \end{pmatrix}$$

$$\mathcal{F}_{hom} : \begin{pmatrix} \textbf{Pointed Smooth Maps} \to \textbf{Linear Maps} \\ (f,p) \to df_p \end{pmatrix}$$

3.3.3 de Rham Cohomology

Let's now go back to two previous categories, namely $\mathbf{Man}_{\mathrm{Smooth}}$ and $\mathbf{Grp}$. Recall that we can define the pull-back for any smooth map $f : M \to N$ between two manifolds M and N denoted by f^*. If $\omega \in \Omega^n(N)$ then we define $f^*\omega \in \Omega^n(M)$ as follows, for $v_1, \ldots, v_n \in T_pM$:

$$(f^*\omega)_p(v_1, \ldots, v_n) = \omega_{f(p)}(df_p(v_1), \ldots, df_p(v_n))$$

The pull-back of any smooth map $f : M \to N$ between two manifolds M and N, denoted by f^* induces an algebra homomorphism

$$f^* : \begin{pmatrix} H_{\mathrm{dR}}(N) \to H_{\mathrm{dR}}(M) \\ [\omega] \to [f^*\omega] \end{pmatrix}$$

Moreover, $(\mathrm{Id}_M)^* = Id_{H_{\mathrm{dR}}(M)}$, and since the pullback of a composition is contravariant the induced homomorphism inherits from it. Hence the rule $\mathcal{F}$ defined below is a contravariant functor.

$$\mathcal{F}_{ob} : \begin{pmatrix} \mathbf{Man}_{\mathrm{Smooth}} \to \mathbf{Grp} \\ M \to H_{\mathrm{dR}}(M) \end{pmatrix}$$

$$\mathcal{F}_{hom} : \begin{pmatrix} \textbf{Smooth Maps} \to \textbf{Group Homomorphisms} \\ f \to f^* \end{pmatrix}$$

3.3.4 The Tangent Bundle

The tangent bundle is defined as

$$TM = \bigsqcup_{p \in M} T_pM.$$

It has the structure of smooth manifold with dimension $2n$ if M has dimension n. If f is a smooth map between two manifolds M and N then the total derivative of f, denoted by df, is a map between tangent bundles.

$$df : \begin{pmatrix} TM \to TN \\ (p, v) \to (f(p), df_p(v)) \end{pmatrix}$$

The composition rule ensure that the rule $\mathcal{F}$ defined below is a covariant functor.

$$\mathcal{F}_{ob} : \begin{pmatrix} \mathbf{Man}_{\mathrm{Smooth}} \to \mathbf{Man}_{\mathrm{Smooth}} \\ M \to TM \end{pmatrix}$$

$$\mathcal{F}_{hom} : \begin{pmatrix} \textbf{Smooth Maps} \to \textbf{Smooth maps} \\ f \to df \end{pmatrix}$$

3.3.5 The Cotangent Bundle

The cotangent bundle of a smooth manifold M is defined as follows

$$T^*M = \bigsqcup_{p \in M} T_p^*M.$$

It is the disjoint union of all dual spaces at each point of the manifold. There is a nice way to interpret the cotangent bundle in physics for instance. Indeed, as explained in [9], "If the manifold M represents the set of possible positions in a dynamical system, then the cotangent bundle T^*M can be thought as representing the set of possible positions and momenta. For example, this is a way to describe the phase space of a pendulum. The state of the pendulum is determined by its position (an angle) and its momentum. The entire state space looks like a cylinder". The cylinder $S^1 \times \mathbb{R}$ is the cotangent bundle of the circle (it is the trivial line bundle over S^1).
For any manifold M of dimension n the cotangent bundle of M has a structure of smooth manifold of dimension $2n$. Later in this report we will see that the cotangent bundle has a natural symplectic manifold structure, hence we will be able to take a look at the cotangent bundle within the Symplectic Manifold category. It is also a smooth vector bundle over M of rank n with its standard projection map $\pi : T^*M \to M$ and the natural vector space on each fiber. Let $p \in M$, consider a chart (U, ϕ) with $p \in U$ and $\phi = (x^1, \ldots, x^n)$. We define the following local frame

$$dx^i : \begin{pmatrix} U \to T^*M|_U \\ p \to dx_p^i \in T_p^*M \end{pmatrix}$$

where $dx_p^i(\frac{\partial}{\partial x_j}|_p) = \delta_{ij}$ (here we make the identification $T_p\mathbb{R}$ and $\mathbb{R}$). The family $dx^1|_U, \ldots, dx^n|_U$ is a local frame over U (the sections being smooth). They are called coordinate covector fields. Since being trivializable over an open set is equivalent to admitting a local frame over U, the sections of the frame being smooth proves that the cotangent bundle is a smooth vector bundle of rank n.
From now on we will consider the following categories:

- **VB** : The objects are smooth vector bundles and the morphisms are smooth bundle homomorphisms,
- $\mathbf{Man}_{\text{Diff}}$: The objects are smooth manifolds and the morphisms are diffeomorphisms.

Proposition 3.1: *Suppose $F : M \to N$ a diffeomorphism, and let $\mathcal{T}^*F : T^*N \to T^*M$ be the map whose restriction to each T^*_qN is $dF^*_{F^{-1}(q)}$, then $\mathcal{T}^*F$ is a smooth bundle homomorphism.*

Proof: If we denote by π_M and π_N the standard projections of T^*M and T^*N, we have to show that the following diagram commutes.

$$\begin{array}{ccc} T^*N & \xrightarrow{\mathcal{T}^*F} & T^*M \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ N & \xrightarrow{F^{-1}} & M \end{array}$$

If we denote by $(V, \psi = (y^1, \dots, y^n))$ a chart at $q \in N$, the coordinate covector fields $dy^1, \dots, dy^n$ allow us to write an element of $\omega \in T^*_qN$ as

$$\omega = \xi^i dy^i|_q.$$

Hence

$$F^{-1} \circ \pi_N(\omega) = F^{-1}(q),$$

$$\pi_M \circ \mathcal{T}^*F(\omega) = \pi_M \circ dF^*_{F^{-1}(q)}(\omega) = F^{-1}(q)$$

since $dF^*_{F^{-1}(q)}(\omega) \in T^*_{F^{-1}(q)}M$. Also $dF^*_{F^{-1}(q)} : T^*_qN \to T^*_{F^{-1}(q)}M$ is a linear map, so $\mathcal{T}^*F$ is linear on each fibers. Lastly, F is assumed to be a diffeomorphism so F^{-1} is smooth. We conclude that $\mathcal{T}^*F$ is a smooth bundle homomorphism.

Proposition 3.2: *The assignment $M \mapsto T^*M$, $F \mapsto \mathcal{T}^*F$ is a contravariant functor from $\boldsymbol{Man}_{Diff}$ to $\boldsymbol{VB}$. It is called the cotangent functor.*

Proof: Let us denote such a rule by $\mathcal{F}$. We just saw that if $F \in \text{Hom}_{\mathbf{Man}_{\text{Diff}}}(M, N)$, then $dF^* \in \text{Hom}_{\mathbf{VB}}(\mathcal{F}(N), \mathcal{F}(M))$. We need to show that

$$\forall M \in \text{Ob}(\mathbf{Man}_{\text{Diff}}), \mathcal{F}(\text{Id}_M) = \text{Id}_{\mathcal{F}(M)}$$

$$\forall (f, g) \in \text{Hom}_{\mathbf{Man}_{\text{Diff}}}(M, N) \times \text{Hom}_{\mathbf{Man}_{\text{Diff}}}(P, M), \mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Let M be a smooth manifold. Id_M is indeed a diffeomorphism as it is smooth and involutive. Moreover, for any element $p \in M$, $d\text{Id}_M = \text{Id}_{TM}$. Hence, $d\text{Id}^*_M = \text{Id}^*_{TM}$. Or, for any $\omega \in T^*_pM$ and $v \in T_pM$, $d\text{Id}^*_M(\omega)(v) = \omega(d\text{Id}_p(v)) = \omega(v)$, i.e. $d\text{Id}^*_M = \text{Id}_{T^*M}$. This proves the first statement. The second statements follows from the fact that the differential is covariant and the pull-back is contravariant. For two maps $G : M \to N$ and $F : N \to P$. For $\omega \in T^*_pP$ and $v \in T_pP$, we have

$$\begin{aligned} \mathcal{T}^*(F \circ G)(\omega)(v) &= \omega(d(F \circ G)^*_{(F \circ G)^{-1}(p)}(v)) \\ &= \omega((dF_{F^{-1}(p)} \circ dG_{G^{-1} \circ F^{-1}(p)})^*(v)) \\ &= \omega(dG^*_{G^{-1} \circ F^{-1}(p)} \circ dF^*_{F^{-1}(p)}(v)) \\ &= \mathcal{T}^*G \circ \mathcal{T}^*F(\omega)(v), \end{aligned} \tag{1}$$

hence $\mathcal{T}^*(F \circ G) = \mathcal{T}^*G \circ \mathcal{T}^*F$. We finally conclude that $\mathcal{T}^*$ is a contravariant functor. The following section will introduce symplectic manifolds and present the category of symplectic manifolds. The aim will be to present a functor from the category of smooth manifolds equipped with diffeomorphisms to the category of symplectic manifolds equipped with symplectomorphisms, for which a smooth manifold M is mapped to T^*M.

3.3.6 Smooth Functors

It is possible to construct vector bundles out of old ones. This is because one can construct new vector space out of old ones via classic operations on vector spaces.

Example 3.3 The Whitney Sum:
If V and W are vector space then $V \oplus W$ is a new vector space. Then, if $E \to M$, $F \to M$ are two vector bundles, we can define $(E \oplus F)_x = E_x \oplus F_x$, yielding a collection of vector spaces out of which it is possible to construct a vector bundle. If $\{g^\alpha_\beta\}$, (resp. $\{h^\alpha_\beta\}$) are the transition functions of E (resp. F), then we can construct a vector bundle, for which the fibers are $E_x \oplus F_x$ and the transitions functions are $\left\{\begin{pmatrix} g^\alpha_\beta & 0 \\ 0 & h^\alpha_\beta \end{pmatrix}\right\}$. This can be generalized to many other operations : E^*, $E \otimes F$, $\bigwedge^k E$,$\bigwedge^k E^*$.... As presented in [2], "The framework which justifies why linear algebraic constructions carry through to vector bundles uses the language of categories.". The relevant category in this framework is the following:

- $\mathcal{V}$, the category whose objects are finite dimensional real or complex vector spaces and whose morphisms are linear isomorphisms.

We define in this precise case what a binary functor is:

Definition 3.4 A *binary functor* is a rule $\mathcal{F} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ such that:

- To each pair of vector spaces V, W it assigns a vector space $\mathcal{F}(V, W)$
- To each pair of isomorphisms f, g it assigns an isomorphism $\mathcal{F}(f, g)$.

And also such that it is functorial in each of its variable :

- $\mathcal{F}(\mathrm{Id}, \mathrm{Id}) = \mathrm{Id}$
- $\mathcal{F}(f \circ f', g \circ g') = \mathcal{F}(f, g) \circ \mathcal{F}(f', g')$.

Definition 3.5 A *smooth (binary) functor* is a functor $\mathcal{F}$ such that $\mathcal{F}(f)$ (resp.$\mathcal{F}(f, g)$) depends smoothly on f and g.

As explained in [2], 'An intuitive and less intrinsic way to think of smooth functors is that bases for V and W give rise to a basis for $T(V, W)$ and smooth changes on the bases of V and W causes the basis of $T(V, W)$ to change smoothly.' And this relates to our construction in the Whitney sum. Indeed, local frames $\{s^E_1, \ldots, s^E_k\}$ and $\{s^F_1, \ldots, s^F_l\}$ on U give rise to the local frame

$$\{(s^E_1, 0), \ldots, (s^E_k, 0), (0, s^F_1), \ldots, (0, s^F_l)\}.$$

The change of basis between $\{s^{\alpha,E}_1, \ldots, s^{\alpha,E}_k\}$ (resp. $\{s^{\alpha,F}_1, \ldots, s^{\alpha,F}_l\}$) and $\{s^{\beta,E}_1, \ldots, s^{\beta,E}_k\}$ (resp. $\{s^{\beta,F}_1, \ldots, s^{\beta,F}_l\}$) is given by the smooth transition function g^α_β (resp. h^α_β). And change of basis on the new frame is given by

$$\begin{pmatrix} g^\alpha_\beta & 0 \\ 0 & h^\alpha_\beta \end{pmatrix}$$

which is smooth because g^α_β and h^α_β are smooth. Note that we can extend a binary functor to a k-functor (defined similarly with k variables instead of two), hence generalize this point of view to many operations on vector spaces.

4 The Symplectic Cotangent Functor

4.1 Symplectic Manifolds

In this section we give the basic definitions from symplectic geometry.

Definition 4.1 Let M^{2d} be a differentiable manifold of even dimension $2d$. A *symplectic structure* on M is a 2-form ω that is

- *closed* i.e. $d\omega = 0$,
- *non-degenerate*: for $v, w \in T_xM$, if $\omega(v, w) = 0$ for all $w \in T_xM$, then $v = 0$.

A pair (M^{2d}, ω) is called a *symplectic manifold.*

Definition 4.2: Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A differentiable map $f : M_1 \to M_2$ is called *symplectic* if the pullback of ω_2 under f is equal to the symplectic structure on M_1:

$$f^*\omega_2 = \omega_1.$$

A symplectic diffeomorphism is called a *symplectomorphism.*

Example 4.4: Let $\mathbb{R}^{2n}$ be the Euclidean space of dimension $2n$ and let $(x^1, \dots, x^n, y^1, \dots, y^n)$ be the standard coordinates. We define the 2-form

$$\omega = \sum_i^n dx^i \wedge dy^i.$$

This form is symplectic, indeed

$$d\omega = d\sum_i^n dx^i \wedge dy^i = \sum_i^n d(dx^i \wedge dy^i) = 0$$

and non-degenerate because its value at each point of $\mathbb{R}^{2n}$ is the symplectic 2-tensor which is non-degenerate.

Example 4.5: Another example of a symplectic manifold is the sphere S^2. Let ω be a smooth non-vanishing 2-form. Then ω is closed because the differential of a 2-form is a 3-form which, defined on a 2-dimensional object, is zero and it's non-degenerate because in two dimensions every non-vanishing form is non-degenerate. This argument can be generalized and we can show that actually every orientable smooth manifold of dimension two admits a symplectic structure.

Lemma 4.6: *The collection **Symp** for which objects are symplectic manifolds and morphisms are symplectomorphisms is a category.*

<u>Proof of Lemma 4.6</u> We can indeed compose morphisms because if $f : (M_1, \omega_1) \to (M_2, \omega_2)$ and $f : (M_2, \omega_2) \to (M_3, \omega_3)$ are symplectomorphisms then $h = g \circ f : (M_1, \omega_1) \to (M_3, \omega_3)$ since $(g \circ f)^*\omega_3 = f^* \circ g^*\omega_3 = f^*\omega_2 = \omega_1$. Composition is associative as classical composition is associative. Notice that the identity map between two symplectic manifolds is a symplectomorphism because the identity between two manifolds induces the identity on forms hence, if (M, ω) is a symplectic manifold, then $\mathrm{Id}_M^*\omega = \omega$. This achieves to prove that **Symp** is a category.

4.2 The Cotangent Functor

The aim of this section is to present the category of symplectic manifolds endowed with symplectomorphisms and to show the following invariance result.

Theorem 4.7: *The canonical symplectic form on the cotangent bundle is invariant under diffeomorphisms in the following sense: Suppose Q and $\tilde{Q}$ are smooth manifolds and $F : Q \to \tilde{Q}$ a diffeomorphism. Let $\mathcal{T}^*F : T^*\tilde{Q} \to T^*Q$ be the map described in section 3.3.5. $\mathcal{T}^*F$ is a symplectomorphism when both $T^*Q, T^*\tilde{Q}$ are endowed with their canonical symplectic forms.*

This result yields the following corollary:

Corollary 4.8: *The assignment $M \mapsto T^*M$, $F \mapsto \mathcal{T}^*F$ is a contravariant functor from* $\boldsymbol{Man}_{Diff}$ *to* ***Symp***.

There is a natural 1-form λ on the total space of T^*M called the **tautological 1-form** that we can obtain from the standard projection, for this purpose we follow the construction made by John M. Lee in [1]. A point in T^*M can be represented as a covector $\phi \in T_q^*M$ for some $q \in M$ that we denote by (q, ϕ). Recall that the standard projection is smooth and is represented by $\pi(q, \phi) = q$. The pointwise differential of π yields a linear map

$$d\pi_{(q,\phi)} : T_{(q,\phi))}(T^*M) \to T_qM$$

As explained in [3], in physics, "M corresponds to configuration space and T^*M to phase space. The point q corresponds to the (generalized) position of a particle and the covector $p \in T_q^*M$ to its (generalized) momentum." The pullback of this map yields a linear map

$$d\pi^*_{(q,\phi)} : T_q^*M \to T^*_{(q,\phi)}(T^*M).$$

Hence we can define the following 1-form $\lambda \in \Omega^1(T^*M)$

$$\lambda_{(q,\phi)} = d\pi^*_{(q,\phi)}\phi \in T^*_{(q,\phi)}(T^*M).$$

Proposition 4.9: *Let M be a smooth manifold. The tautological 1-form is smooth, and $\omega = -d\lambda$ is a symplectic form on the total space of T^*M.*

To prove this, we need the following lemma:

Lemma 4.10: *Let $\pi : E \to M$ be a smooth vector bundle of rank k. L et (U, ϕ) be a smooth chart on M with coordinate (x^i), and suppose there exists a smooth local frame (σ_i) for E over U. Define $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^k$ by*

$$\tilde{\phi}(v^i\sigma_i(p)) = (x^1(p), \dots, x^n(p), v^1, \dots, v^k).$$

Then $(\pi^{-1}(U), \tilde{\phi})$ is a smooth coordinate chart over E.

Proof of Lemma 4.10: Every smooth local frame for a smooth vector bundle is associated with a smooth local trivialization. Let Φ denote the smooth local trivialization over U that corresponds to the local frame (σ_i), it is defined by

$$\Phi(v^i\sigma_i(p)) = (p, (v^1, \dots, v^n)).$$

Then $\tilde{\phi} = (\phi \times \mathrm{Id}_{\mathbb{R}^k}) \circ \Phi$, this proves that $\tilde{\phi}$ is a smooth map.

Proof of Proposition 4.9: Let $(U, \phi = (x^i))$ be a chart. We saw that (dx^i) forms a smooth local frame for T^*M over U. Now, Lemma 4.10 ensures that the map from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\xi_i dx^i|_p \mapsto (x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n)$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) the natural coordinates for T^*M associated with (x^i). As shown in [1], λ has the following natural coordinates representation

$$\lambda_{(x,\xi)} = d\pi^*_{(x,\xi)}(\xi_i dx^i) = \xi_i dx^i.$$

Hence, λ is smooth because it is linear in its components. Furthermore, ω is exact and therefore closed. We need to verify that it is non-degenerate. We have

$$\omega = -d\lambda = \sum_{i=1}^{n} dx_i \wedge d\xi_i$$

which, under identification, corresponds to the standard symplectic 2-form in $\mathbb{R}^{2n}$.

Note that there is a reciprocity theorem, called the Darboux Theorem that won't be proved here, a proof can be found in [1]. The result is the following:

Theorem 4.11: *Let (M, ω) be a $2n$-dimensional symplectic manifold. For any $p \in M$ there are smooth coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ centered at p in which ω has the coordinate representation:*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

The symplectic form ω defined in proposition 4.9 is called the **canonical symplectic form** on T^*M. Now we have the tools to prove Theorem 4.7, since we have a symplectic structure on the cotangent bundle.

Proof of Theorem 4.7: Suppose Q and $\tilde{Q}$ are smooth manifolds and $F : Q \to \tilde{Q}$ a diffeomorphism. Let $\mathcal{T}^*F : T^*\tilde{Q} \to T^*Q$ be the morphism described in section 3.3.5, and denote by $\omega^Q, \omega^{\tilde{Q}}$ the canonical symplectic forms on T^*Q and $T^*\tilde{Q}$, respectively. Denote $\Phi = \mathcal{T}^*F$, we need to show that $\Phi^*\omega^Q = \omega^{\tilde{Q}}$. Notice that it suffices to show that $\Phi^*\lambda^Q = \lambda^{\tilde{Q}}$, so we compute $\left(\Phi^*\lambda^Q\right)_{\tilde{q},\tilde{\phi}}$:

$$\begin{aligned}
\left(\Phi^*\lambda^Q\right)_{\tilde{q},\tilde{\phi}} &= \lambda^Q_{\Phi(\tilde{q},\tilde{\phi})} d\Phi_{\tilde{q},\tilde{\phi}} \\
&= dF^*_{F^{-1}(\tilde{q})}(\tilde{\phi})(d\pi_{\Phi(\tilde{q},\tilde{\phi})} d\Phi_{\tilde{q},\tilde{\phi}}) \\
&= dF^*_{F^{-1}(\tilde{q})}(\tilde{\phi}) d(\pi \circ \Phi)_{\tilde{q},\tilde{\phi}} \\
&= dF^*_{F^{-1}(\tilde{q})}(\tilde{\phi}) d(F^{-1} \circ \tilde{\pi})_{\tilde{q},\tilde{\phi}} \\
&= \tilde{\phi}(dF_{F^{-1}(\tilde{q})} dF^{-1}_{\tilde{q}} d\pi_{\tilde{q},\tilde{\phi}}) \\
&= \tilde{\phi}(d\pi_{\tilde{q},\tilde{\phi}}) \\
&= \lambda^{\tilde{Q}}_{(\tilde{q},\tilde{\phi})}.
\end{aligned} \tag{2}$$

To go from the third equality to the fourth one, recall that we showed that Φ is a bundle homomorphism between the cotangent bundles (cf proposition 3.1), i.e. $F^{-1} \circ \tilde{\pi} = \pi \circ \Phi$. This proves that $\Phi^*\lambda^Q = \lambda^{\tilde{Q}}$. Furthermore, the property of the pullback ensures that

$$\Phi^*(d\lambda^Q) = d(\Phi^*\lambda^{\tilde{Q}}),$$

hence

$$\Phi^*\omega^Q = \omega^{\tilde{Q}}.$$

This proves Theorem 4.7 and the corollary follows directly. Notice that it also would have worked to consider local diffeomorphisms instead of global diffeomorphisms by modifying our functor. If we define $\mathcal{T}^*$ to act on local diffeomorphism as follow:

$$T^*f(x, \alpha) = (f(x), \alpha(df(x)^{-1}))$$

where $\alpha \in T^*_xQ$ and $x \in Q$. This yields, by very similar computation, a covariant functor between the category of smooth manifolds equipped with local diffeomorphisms $\mathbf{Man}_{\text{Loc}}$ to symplectic manifolds **Symp** equipped with symplectomorphisms.

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