COLLECTIVE MASS EDUCATION

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Foreword.

This work has been created by me for many years, collected material and worked in the preparation of the same, but this often interrupted by other work, long time completely put aside and thus the completion of the work has been delayed. To delay it for longer would not be advisable for my age if the work was to appear at all; I also believe that, after repeated return, it may at last dare to appear, not as a perfect work, but as a basis for further development of the doctrine dealt with here. More specifically, the following introductory chapter discusses the task of teaching; and so here only the following general remarks on it may find room.

With the new name under which the teaching occurs here, I do not give it as a completely new doctrine; only that the present state of its development did not yet suggest the need to set it up for itself under a special name. Everywhere science is specializing in the course of its growing development and accordingly demands separating terms of its various areas. Now the most general, most interesting, most meritorious, what has been present of our teaching, in quetelet's "Lettres sur la théorie des probabilités" (1846) and his "Physique sociale." (1869) and, if you like, you can see in him as well the father of the collective theory of measure as in EH Weber that of psychophysics; but one will be able to convince oneself from the pursuit of this work,

In this respect, I assert, from one side, as a principal fruit, and from another as the chief, the mathematical justification and empirical proof of a generalization of Gauss's law of accidental deviations, which confines its limitation to symmetric probability and proportion Smallness of the mutual deviations from the arithmetic mean is raised, and hitherto unknown legal relations occur, the most important of which is found in § 33. In fact, in this generalization, the most general regulator of all relationships in the collective gauge is given as well as in the simple GAUSSian law the regulator of all physical and astronomical precision determinations,

Insofar as the collective gauge is based on a combination of observation and calculation in a mutual relationship, it can count on the exact doctrines. However, the doctrines that claim to have such a name allow for a very different degree of certainty in their results. At the top are mechanics, astronomy, physics; Physiology stands far behind because of the difficulties which oppose the complication and variability of its objects; still further, because of even greater difficulties in this regard, psychophysics. The collective gauge shares difficulties of this kind, without being subject to the same basic difficulties as psychophysics, surpassing them in practical interest, while far outstripping it in philosophical interest.

As to the form and breadth of many accounts, it must be borne in mind that the work is not intended both for subject mathematicians who are already familiar with the fundamental points in consideration here, and for those who are concerned with knowledge and application of the doctrine is, without they already have such prior knowledge.

In the near future, I would like to send a request to computers of the subject to promote our teaching. In the known tables which GAUSSian probability integral of accidental deviations from means (observational error) is usually considered

The argument *t is* only expressed down to two decimals, which is sufficient for the limited use that physicists and astronomers make of it, with the use of interpolation with first and second differences; but for the more widespread use that can be made of it in collective gauge theory, it comes to the same thing as reducing the numerical argument, to which the logarithms belong, to only two or three digits for the many calculations to be performed by logarithms and intermediate provisions only the interpolation wanted to give. So it would be desirable if, in the interest of our doctrine, which incidentally is shared by the psychophysical method of right and wrong cases, there are tables in which *t*at least four decimals [1)] in order to partly spare and partly to facilitate interpolations, and in any case I myself have painfully missed such tables in the execution of this work. Of course, the expansion of the tables would grow with it, but the advantage seemed to me to grow more in line with it. And should not there be an astronomical or statistical institute, which has to have mechanical computing power to take care of the thing! Also, a price task could probably be put on it.

[1] An execution of this table on three decimals of *t,* with limitation of the integral value to four resp. five decimals, can be found in the Appendix § 183.

I. Introduction.

§ 1. Under a collective object (short K.-G.) I understand an object that consists of indefinitely many, randomly varying, copies, which are held together by a species or generic term.

Thus man forms a collective object in the broader sense, man of a certain sex, of a certain age, and of a certain race, one in the narrower sense, as in general what one calls the scope of a K.-G. may change according to the extension of the generic or species term under which it occurs.

The copies of a K.-G. may be spatially or temporally different and hereafter a spatial or temporal K.-G. form. Thus, the recruits of a country or ears of a cornfield as specimens of a spatial K.-G. be valid. Thus, the (mean) temperature of the 1st of January, followed by a number of years in a given place, gives as many copies of a temporal K.-G. Instead of the 1st of January, one can celebrate every other anniversary, instead of a certain day certain month, instead of the temperature set the Barometerstand etc etc and thus become copies of so many temporal K.-G. receive.

Anthropology, zoology, botany have anything at all essential with K.-G. because it can not be a characteristic of individual specimens, but only that which belongs to an aggregate of them, which from one point or another is summed up as a genus or species in greater or lesser extent. Meteorology, according to the examples just cited, offers numerous examples in its non-periodic weather phenomena; and even in artistry one can speak of such as books, business cards are among them.

The copies of a K.-G. Now, on the one hand, qualitatively, on the other hand, quantitatively, ie, in measure and number, are determined, and only the latter determinateness is involved in collective measurement. A K.-G. In fact, in terms of its quantitative certainty, makes the same claims as a single object; only that in a certain (admittedly only certain) respect the parts of the single object are represented by the copies of the K.-G. be represented. Does it apply z. Recruits of a given country, for example, ask themselves: how great are the recruits on average, how much do the individual measures vary by their mean, what are the largest and the smallest, how are the measures of the recruits according to these provisions in the individual years? , as in different countries among each other. Such and related, Questions to be considered later can be found at each K.-G. raise; and insofar as a spatial object has different parts and dimensions to be distinguished, it can be particularly raised to each of these parts and dimensions, and these are therefore considered as special K.-G. treat, so skull, brain, hands, feet of a person, height, weight, volume of the whole person or given parts of the person; but also quantitative relationships will come into question, just as the ratios of the average height, breadth, and length of the skull take up a special interest in comparing the people of different races. and these in so far as special K.-G. treat, so skull, brain, hands, feet of a person, height, weight, volume of the whole person or given parts of the person; but also quantitative relationships will come into question, just as the ratios of the average height, breadth, and length of the skull take up a special interest in comparing the people of different races. and these in so far as special K.-G. treat, so skull, brain, hands, feet of a person, height, weight, volume of the whole person or given parts of the person; but also quantitative relationships will come into question, just as the ratios of the average height, breadth, and length of the skull take up a special interest in comparing the people of different races.

§2. On all these individual questions, however, a more general, the most important one, which can be dealt with in this doctrine at all and accordingly will be dealt with below, raises the question of the law of how the copies of a K.-G. distribute according to size and number. The expression distribution, however, is the definition of how the number of copies of a given K.-G. with their size changes. For each K.-G. existing in a larger number of copies. The smallest and largest specimens, extremes for short, are the rarest, most often those of a certain medium size. But is not there a general, all or at least most K.-G. applicable law of dependence of number on the size of specimens? In fact, such will be set up,

From the outset, of course, one can doubt that in the extraordinary diversity of the K.-G. legal distributional relations are to be found in a certain generality for it at all. Meanwhile, since according to the terms of K.-G. In any case, the general probability laws of chance - and every mathematician knows that such are - find application. In fact, the distribution ratios of K.-G. Generally dominated by such, while the same laws of probability in physical and astronomical measures only marginally for the safety determination of the acquired mediocrities come into consideration, here play a very different and much more insignificant role than in the gauge of K.-G .. Insofar however the accident under certain, for the various K.-G. plays different, external and internal conditions, let through all coincidences through, the various K.-G. distinguish by characteristic, derived from their distribution ratios constants. These are they in which the determinateness of them rest against one another; and these should be consulted in consideration of the general laws of probability. From this point of view, the arithmetic mean of the specimens has always been envisaged, and diligence in its determination by the various K.-G. turned, besides probably extremes, more seldom considered average deviation from means. But as important as these determinants are and will always be, they have so far been taken too unilaterally, while others, in principle no less important, are disregarded.

Insofar as the treatment of K.-G. According to the totality of the previous relations, it is subject to different points of view and carries different modes of determination than are taken into consideration in physical and astronomical measures. Thus the K.-G., or, collectively, Kollektivmaßlehre, can be specially set up and treated as a doctrine of its kind and this will be done as follows.

Since in our concept the K.-G. If the concept of a random variation of the specimens comes in, one can first of all desire a definition of chance and an explanation of its essence. The attempt to give it from a philosophical point of view would, however, be of little use for the following investigation. It must suffice here to indicate the factual aspect, on the basis of the following, of more negative than positive character. By a random variation of the specimens I mean one which is just as independent of arbitrariness as of determination of size, and of a laws of nature governing the proportions of relations between them. If one or the other shares in the provisions of the objects, then only the independent changes happen by chance.

This does not deny that, from the most general point of view, there is no coincidence, in that the size of each individual specimen can necessarily be regarded as definite by the existing laws of nature under the existing conditions. But we speak of coincidence as long as we can neither ascend to a derivation of the individual determinations from such general laws, nor be able to deduce them from the present facts. Insofar as it is the case, the coincidence ceases, and ceases or is disturbed by the applicability of the laws to be presented here.

II. Preliminary overview of the most important points which are found in the investigation of a K.-G. eligible terms and related terms.

§3. The following compilation will serve to give more prominence to the extent and character of the investigations we are about to deal with, and to anticipate most of the terms to be used in advance; but a more detailed discussion of these points is reserved for the following chapters.

In the random order in which the copies of a K.-G. It would not be possible to obtain an overview of their relationships in terms of quantity and number, nor would it be possible to work methodically on them if their measurements , generally indicated by *a* , are given in the same random order in which they are received and in one wanted to keep so-called Urliste wanted; so they are to be sorted according to their size and listed in a table, so-called distribution panel. Did they now there are no large number of copies of an object, each is *a* , or at least most will *a* appear only once in the table, and the size of distances between successive *a* very irregular change; but in many objects, that is, of which there are many specimens, as they are chiefly to be presupposed for the following, if not all but many, or most of the *a,* which the scale and the estimate yields, will occur more or less often, and then If one arranges the distribution table in such a way that in a column of *a* every *a* is only listed once, but in an enclosed column of *z* the number *z is* indicated, how often it occurs. The total number of *a*, which enter into a distribution table, of course, agrees with the sum $\sum$ z *,*which is obtained by adding up all *z of* the table, and is denoted by *m* by me .

The preparation of such a table is, so to speak, the first step that can be taken when working on a variety K.-G. from the original list.

A second step is this: that the, with *A* determined to be designated, the arithmetic mean of the individual measurements and the positive and negative deviations, the number *z* of course with the deviating *a* match.

For this purpose, however, as a starting point of the deviations instead of *A* , some other values which can be derived with mathematical certainty from the distribution table can serve; and by any other choice in this regard, new relationships emerge which will be discussed later. In general, I call values, which are used for the development of such relations as initial values of the deviations, principal values and denote them by *H, of* which *A*is only a special case, which has been taken into account in the treatment of K.-G. but it has an arbitrary limitation of collective theory, as will be readily apparent from the following remarks. In general, I name deviations, of which principal values they may be made dependent, collective deviations.

§ 4. It is easy to convince oneself of the following circumstances. A larger m in the distribution board of a K.-G. the more regular is the course of the z associated with the a , and the more certain are the laws of which we shall have to speak. The ideal case would be that an infinite m would have, where you have a very regular course of *such* would have to be expected and a very exact fulfillment of the relevant regularities, after which also ideal conditions and regularities, as it would give an ideal panel and empirical, which consist in more or less close approximations.

All probability laws of chance at all, and the distribution laws of K.-G. such are common, that their observance is to be expected all the more confident, to a larger number of cases they relate, but have an almost ideal validity only in the case of an infinite number of cases, which does not exclude that already in the case of a number of cases that are empirically well-ascertained, the confirmation of the laws concerned takes place in close approximation. Insofar as in any case in reality only with K.-G. from a finite number of specimens representing as many cases; I designate the deviations, which take place on account of finiteness of the number of copies of the ideal legal provisions, as unimportant, and inasmuch as they go indifferently to one side or the other, as evoked by unbalanced contingencies, while I designate the determinations in force for an infinite number of cases, our case of specimens, as essential or normal. The general feature of the insignificance of a provision is that it disappears the more the number of cases, resp. Specimens, in compliance with the conditions which the term of K.-G. determine, magnified, so that one can assume that it would disappear altogether in an infinite number of cases; according to which, in our case, only very numerous objects are suitable for the investigation of the laws in our case. for the presumption of an infinite number of cases, our case of specimens, current provisions as essential or normal. The general feature of the insignificance of a provision is that it disappears the more the number of cases, resp. Specimens, in compliance with the conditions which the term of K.-G. determine, magnified, so that one can assume that it would disappear altogether in an infinite number of cases; according to which, in our case, only very numerous objects are suitable for the investigation of the laws in our case. for the presumption of an infinite number of cases, our case of specimens, current provisions as essential or normal. The general feature of the insignificance of a provision is that it disappears the more the number of cases, resp. Specimens, in compliance with the conditions which the term of K.-G. determine, magnified, so that one can assume that it would disappear altogether in an infinite number of cases; according to which, in our case, only very numerous objects are suitable for the investigation of the laws in our case. in compliance with the conditions which the term of the K.-G. determine, magnified, so that one can assume that it would disappear altogether in an infinite number of cases; according to which, in our case, only very numerous objects are suitable for the investigation of the laws in our case. in compliance with the conditions which the term of the K.-G. determine, magnified, so that one can assume that it would disappear altogether in an infinite number of cases; according to which, in our case, only very numerous objects are suitable for the investigation of the laws in our case.

Even with a small *m* , however, the insignificance of a determination proves the fact that on repeating the determination with the same small *m* size and direction of the determination changes undetermined from getting new copies of the same object, whereas in essentiality the same on average a majority of repetitions a certain size result and the stronger the number of repetitions, and the *m of* each individual, the stronger the certainty of a particular direction .

We speak of a symmetrical distribution of values with respect to a given principal value *H,* when any deviation of *a* positive-of *H* equally large negative deviation of another *a* of *H*corresponds to, so that equal strength on both sides of *H* deviating *a* equal *z*To belong. At a K.-G. From a finite number of specimens, it can not be expected at all, because of unbalanced contingencies, to find a completely symmetrical distribution with respect to any principal value, and of course a symmetrical distribution can not exist at the same time with respect to several principal values; but it is an important object of the investigation whether a principal value can be found with respect to which the distribution approaches the more symmetrical the more one has the *m of* the K.-G. magnified, in such a way that with infinite *m* one could presuppose a truly symmetrical distribution as attained, in which case, since an infinite *m*is not available, but can speak of a symmetrical probability of deviations.

§5. But even from a point of view different from the previous point of view, one can distinguish an ideal distribution panel from an empirical and dependent therefrom ideal and empirical results. In the measurements of the specimens, one can not go beyond certain limits of accuracy as given by the scale scale and the estimate in between. You can z. B. still millimeters, even tenths of a millimeter, still hundredths of a millimeter but not beyond distinguish. For the one who distinguishes only millimeters, all individual measures that are within the limits of one millimeter are indistinguishable, and thus he relates all the *z* copies, which are actually distributed over a whole interval of 1 mm, to a single value *a*, which forms the middle of this interval. Be well *i* heard the still discernible difference in the extent to which *such* each *a* empirical panel actually the whole interval of the size *i* between $a - {}^{1}/_{2}\, i$ and $a + {}^{1}/_{2}\, i$ on, while it is according to the empirical panel so exceptions and in the utilization of the same is usually taken as if the amount falling *a* itself *z* times would occur. However, if the measurement and estimation were ideal, that is, to the limit of accuracy, *i* to descend to an infinitesimally small value [1)] , the differentiated *a of* the panel will multiply hereby, but their *z* will decrease correspondingly; hereby the ideal table deviate from the empirical one.

[1] An infinitesimally small value, here in the sense of calculus, is not to be confused with zero, but, although decreasing below any executable magnitude and indeterminable in absolute magnitude, it can be calculated by its relations to other infinitesimal values.

Now, where the empirical *i is* very small, the results of the empirical table, insofar as they concern the size and relationships of the principal values and principal deviations derived therefrom, do not differ materially from those of the ideal ones; but the difference remains generally to be considered, and will later find this consideration where it comes in considerable consideration. Empirical terms and conditions in which he is not required taken into account, but it is considered as if really *, for* every *a* this *a* very zukäme, I call raw, those where he is as far as possible take into account sharp.

§ 6. In any case, one must now seek to ascend from the results of the empirical table to the ideals of the ideal table, herewith from insignificant to essential, from raw to sharp, to which belongs a corresponding elaboration of the distribution tables.

In this regard, there is a difference between primary and reduced panels. By primary tables I mean those which are obtained directly by the order of the measures from the original list, and hereby present the same data of experience as these, only ordered. Reduced panels I'm those in which the *z* for larger Maßintervalle are discriminated as in the primary panels, and while the total for the same size throughout the whole panel, the *z* but these larger intervals the centers thereof, as reduced *a,* are added inscribed , with the advantages, thereby a more regular course of *z*in the blackboard and to get a more suitable document for bills, if not without conflict with a drawbacks because of enlargement of the *i,* to come back to later. A more in-depth discussion of the way in which the primary and reduced panels are arranged in Chapters VII and VIII is discussed, with the possibility of various reduction stages and reduction situations being discussed.

§ 7. In each not too irregular primary or by reduction regularly made blackboard one finds the following.

The smallest *z* are found after the two limits of the table, according to which, as previously touched, the smallest and largest *a are* the least common, but the largest *z* generally in a middle part of the table. The maximum *z* falls on some *a* in this middle part, where on both sides the *z* to the extremes continuously, albeit with insufficient reduction here and there interrupted by irregularities decrease. The value *a of* a not too irregular primary or reduced distribution table to which the maximum *z*I shall call the densest value of the panel, or empirically the most dense value of the object, which, of course, can only be considered as approximation to the ideal most dense value which one would obtain at infinitely large *m* and infinitely small *i* , but which is no less of *A* The table applies, but deserves special attention even as such rapprochement and offers the basis for a closer approximation by calculation in a manner to be considered later. Be it empirical or ideal, in this or that approximation, I generally call it *D.*

One might believe that the densest value significantly, that is, from a very large, strictly speaking an infinite *m* and, strictly speaking, with a very small infinitesimal *i,* determined, would coincide with the arithmetic means, and in fact soft in the majority of K. -G. both after determination from large *m* and small *i*little enough of each other that one can be inclined and so far in fact has held that the remaining deviation is merely a matter of unbalanced contingency. It will, however, be one of the most important results of the following study that a substantial deviation between arithmetic mean and densest value is rather the general case, the way that magnitude and direction of this deviation itself are characteristic of various K.-G. are. Insofar as the deviations with respect to both values also comply with different ratios, the empirically denser value *D is to be recognized* as an important principal value to be distinguished from the arithmetic mean *A of the* same panel, namely the output value of collective deviations.

For the two preceding principal values *A, D* , there is another, third, which I shall designate as the central value or center of value, *C* , that is, the value of *a,* which has just as many larger *a* above itself as smaller ones, and in this insight, the series of *a* cuts through the middle. The same thing happens when one says that it is the value that makes the number of positive deviations equal to the number of negative ones. It differs from the arithmetic mean by the two determinations that, while with respect to *A* the sum of the mutual deviations is the same, with respect to *C*the number of mutual deviations is equal, and that while bez. *A is* the sum of the squares of the deviations a minimum, that is smaller than bez. any other initial value is against this bez. *C is* the sum of the simple deviations (the negative while calculated on the absolute value) in the same sense a minimum [2)]. With the addition of this third main value to the two previous ones, new characteristic relationships are once again opening up for the K.-G. of which to speak.

[(2)] I have proved this property of the central value, which was not previously noticed, in a special treatise on the same (about the initial value of the smallest amount of variance; Abhandl. the math phys. Class of the royal Sächs. Society of Sciences; II. Volume, 1878].

In addition to the three principal values mentioned above, others which are mathematically derivable from the table of distributions may serve as initial values of deviations and hereby as principal values, and may be considered partly independent of the previous ones, and partly related to them; but in any case, the previous ones are the most important ones, and I stay with them for the time being. In a later chapter (chapter X), however, I will consider negligibly three other principal values as a divisor value *R* , heaviest value *T* and deviation value *F* , which in any case represent a mathematical interest.

§ 8. An animal is characterized by its inner structure, through the brain, heart, stomach, liver, etc., the size and position of these organs against each other, the feeding and the discharging ways to it. So is a K.-G. characterized by its arithmetic mean, central value, densest value, and otherwise approximate main values, the size and position of these principal values against each other and the deviations thereof; and these values are no less mathematical than those organs in organic connexion. A K.-G. so to speak constitutes a mathematical organism capable of a decomposition, which will be discussed below. And if that does not mean that every object has to claim for the accomplishment of such a decomposition,

To begin with, it may be noted that, however, under a certain condition, the two principal values *D* and *C would* coincide with *A,* and consequently all three would coincide with one another, on the condition that the mutual deviations be. *A had* a symmetrical probability, that is to say, with increasing *m* in the manner of a symmetrical distribution (in the sense above), that at infinite *m* one could regard such as attained. But it will turn out that for K.-G. rather an asymmetric probability of deviations. *A has to* presuppose which according to one with increasing *m*does not approach a symmetrical distribution, but rather a substantially asymmetrical distribution to be brought to a certain law. Yes, apart from the essential coincidence of *D* and *C* with *A,* which can only be regarded as an exception, no value at all can be given for K.-G. find, bez. whose symmetrical probability of deviations would take place on both sides.

If, in the treatment of K.-G., we have hitherto considered only *A,* the deviations thereof, and the extremes, we see not only from the preceding that quite important characteristic relations and differences of objects are disregarded, but rather It will also be shown that a general distribution law of the copies of K.-G. can not be won by this limited treatment.

But it is undisputed that this is due to the fact that the guiding points of the physical and astronomical theory of measurement have been transferred to the collective theory without taking into account two essential differences existing between the two, which motivates the limited mode of treatment of the former doctrine as well the latter is denied. For the former, the arithmetic mean *A of* the observation values of the individual object to be determined according to its dimensions, with the deviations of *A,*of observation errors, the dominant, even basically counting, meaning, since, for reasons known to professional mathematicians and physicists, in the value in relation to which the sum of the squares of the deviations, the error, is the smallest possible Arithmetic means, at the same time sees the value which comes closest to the true value of which it is to be determined, but in the deviations of it finds a means of determining the magnitude by which the true value still participates given probability of one or the other side is missed. So why care in this doctrine for other main values that help and their deviations to fulfill the task of this doctrine nothing! So neither is it of a dense value,*a* , could as well give rise to the derivation of a *D* and *C* ; as the various copies of a K.-G. But it would be pointless to look at it in a special way, and it certainly does not happen.

For the collective theory of measure, however, the point of view which, in principle, allows the arithmetic mean value with its deviations from it in physical and astronomical theory of measure, has no significance whatsoever. All copies of a K.-

G., even if they deviate so much from the arithmetic mean or any other principal values, are equal and true, and a preferential consideration of one before the other from a point of view that is equally vain for all has, of course, no sense , On the other hand, every other principal value according to another relationship has its characteristic and sometimes even practical significance for a K.-G., thereby contributing to its differentiation from other objects.

Secondly, however, according to the symmetrical probabilities which have been postulated or presupposed in the physical and astronomical theory of measurement, the symmetrical probabilities, as proved beyond doubt, are different. of the arithmetic means of observation, on good observation, the three principal values are not essential, but only by unbalanced contingencies of each other, so that the most probable values of the other principal values are found in the arithmetic mean of the observed values, which is to be preferred because of the circumstance cited , remarkably an asymmetric probability of deviations. of the arithmetic mean is to be regarded as the general case, according to which the principal values differ considerably.

Incidentally, it may even be questionable whether, with this postulate, the error of observation is really entirely in the right, a question which, although not essential to us, will be considered later in a special chapter [3] .

[3] [With regard to this question, the second part, chap. XXVIII examines the asymmetry of error series.]

But let us now return to the conditions which are essential for collective measurement.

§ 9. Under elements or determinations of a K.-G. In the analysis of such values I will understand the following terms, some of which have been used earlier.

1) The total number of copies *,* generally designated *m* , of a considered distribution panel.

2) The principal values or output values of deviations generally designated *H* , of which the arithmetic mean *A* , the central value *C* and the densest value *D are* the most important. Since the central value is generally to be found between *A* and *D* , as will be shown later, the three main values above will generally be listed by me in the following order *A, C, D.* Here are a few main values to be taken into account, which are discussed in Chapter X.

The arithmetic mean, determined from the *a of* a primary panel, with A_1*,* from which a reduced one is determined, is designated A_2 ; in accordance with *C.* In *D* no such distinction is made because it because of irregularities to have can be derived related bids primary panels everywhere just from reduced panels, hereby anywhere with D_2 could be described. Against this is. to make a distinction in the way of the derivation. According to the so-called Proportionsverfahren, which gives me the most confidence, derived, I call him D_p, *derived* by the less secure interpolation method, with D_i. The difference between the two ways of proceeding will continue to be discussed.

All the values that fall on the positive side of the principal value to which they are related, I designate with a dash above, all that fall on the negative side, with a dash below, while I count on those who indiscriminately refer to both sides, omitting the dashed lines, according to which a' denotes a value a, which exceeds H, $a_,$ one which is exceeded by *H.*

By Θ I generally mean deviations from any principal value *H;* under $\Theta' = a' - H$ a positive, under $\Theta_, = a_, - H$ a negative, if the negative character of $\Theta_,$ to be maintained; but since in general the negative deviations according to their absolute values, as positive, will be to charge, but rather is to be set $\Theta_, = H - a_,$. After that, with $\sum\Theta' = \sum(a' - H)$ the sum of the positive deviations, with $\sum\Theta_, = \sum(H - a_,)$ that of the negative deviations according to absolute values, with $\sum\Theta = \sum\Theta' + \sum\Theta_,$ the total sum of the deviations. *H*denotes.

3) The main deviation numbers di are the number of deviations Θ of given principal values *H,* which of course coincides with the number of deviating values a, ie the total number is equal to m, irrespective of the nature of the principal values , whereas the number of positive and negative Θσ ισ especially with the nature of the main values changes and as positive generally with m', as negative with $m_,$ be designated. From m' and $m_,$ then the differences are $\pm(m' - m_,)$ and the ratios $m' : m_,$ and $m_, : m'$ depends on which instead m' and $m_,$ may be mentioned, provided from them by consultation of m, the values of m' and $m_,$ follow (see below.).

4) The main deviations and. resulting in mean deviations, ie sums of deviations divided by the number of deviations. The total sum of the deviations in both directions together, at its absolute values as we always believe is expressed by AΘ off individually to both sides, in particular by AΘ ′ and $A\Theta_,$ so that $A\Theta = A\Theta' + A\Theta_,$. Dependent on this are the simple mean deviations or mean deviations [4] :

The total sums of the deviations ∑Θ do not remain the same as the total numbers m according to the principal values, but change no less than the one-sided sums according to the principal values.

[4] In the physical and astronomical error calculation rather than the root mean square root meander from the mean square error

, bez. A, which I refer to, where it refers to, as indicated by the following number 5) as the quadratic mean deviation from the simple one determined above, and denoted by q .

With respect to the arithmetic mean A particular mutual deviation sums $A\theta$ 'and $A\Theta_,$ need the same because this is in terms of this remedy itself, however, the mutual deviation numbers m', $m_,$ bez. of this agent are not equal in general, which carries that the unilateral average deviations $\varepsilon' = A\Theta' : m'$, $\sigma_, = A\theta_, : m_,$ dist. A are generally unequal. The mutually valid $\varepsilon = \sum\Theta : m$ is not to be found or determined as a simple means between ε' and $\varepsilon_, = \frac{1}{2}(\varepsilon' + \varepsilon_,)$, as I erroneously stated in an American treatise on the measures of the recruits (by elliott [5)]) not on it

returns; but this is only the case when in the middle drawing from ε' and $\varepsilon_,$ of the considered weights which by virtue of them m' and $m_,$ from which they are received, send, hereafter sets:

which is attributed to $\varepsilon = \sum\Theta : m$ after the following simple consideration . Since the product of an agent from variations in the number of which is equal to the sum of the deviation, then $m'\varepsilon' = A\Theta'$ and $m_, \ \sigma_, = A\Theta_,$ so $m'\varepsilon' + m_, \ \varepsilon_, = A\Theta' + \sum\Theta_, = \sum\Theta_,$ on the other hand

$$m' + m_, = m.$$

5) [EB elliott, On the military statistics of the United States of America; Berlin 1863. International statistical congress at Berlin.]

The greater the mean deviation ε with respect to a principal value, the more the averagely the individual values a deviate from it, or the more they fluctuate on average by the same. Apart from the absolute size of ε but also his relationship comes to the H, followed by ε refers, so $\varepsilon : H$ into account what I call the relative variation. The mean and relative mean variation for a given mAlthough not proportional to the different main values; yet, generally speaking, they increase and decrease in such a degree that an object which varies greatly or faintly with respect to a certain principal value can be regarded as strong or weakly fluctuating with respect to the other principal values, and thus without consideration of a particular one Main value of strong and weak on average or relatively unsteady objects.

After this, the following remark. The size of the simple sum $\sum\Theta$ and the simple μεαν εppop $\varepsilon = \sum\Theta : m$ with respect to the arithmetic mean A is not entirely independent of the number m *of* the values $a,$ from which the A *in* question is derived, but increases somewhat with increasing *m* ; but one can obtain the values ∑Θ and ε bez obtained at any finite *m* . *A*by multiplication with the normal case that they bez. an *A* of an infinite number of *a* obtained what I call the correction due to the finite *m* call [6] . Now, ωηιλε ∑Θ and *ε* =∑Θ : *m are* the uncorrected values, I δενοτε the corrected values with $∑Θ_c$ and $ε_c$:

and .

However, only at very small *m* are the corrected values significantly different from the uncorrected ones, and since we generally have to deal with large *m,* whereas 1 disappears noticeably, I content myself in performing the elements generally with indication of the common, ie uncorrected values ∑Θ *,* ε , from which, with the help of the always known *m,* the corrected values can be easily found when it is necessary to do so. A corresponding remark is undisputed for the deviation sums and mean deviations. other main values than *A* are valid, even if the direct investigation in this respect up to now only on the deviations of *A* has extended. But the less a matter of giving and using the elements obtained in a given finite*m* , the corrected values are to be preferred; as not only the variance sums and mean deviations bez. but also the deviations of the principal values themselves from each other are under the influence of the same finite *m* , the relations of which, therefore, would not change by the common correction. In examining the laws of distribution, however, it is more important for us to arrive at these ratios than at absolute values. However, where one wants to go to such, one has with regard to correction of the one-sided values ∑Θ *',* ∑Θ **,** and ε ', E **,** find the note instead that they do not respectively by and , but like that of AΘ and ε by

must be done because otherwise by adding the corrected values AΘ *',* AΘ **,** the corrected sum AΘ would not find. Also, the rational point of view lies in the fact that the deviation sums of each page as members of the total deviation sum must be influenced jointly by the size of their *m* .

[6] As is well known, GAUSS has already for the sum of the error squares $∑Θ^2$ bez. *A* and the derived from it, so-called quadratic mean error

determines the correction because of the finite *m* ; according to which the former is done by multiplication by *m:* (*m* - 1), the latter being in accordance with our correction of the simple mean error. The theoretical derivation and empirical προοφ οφ our correction of ∑Θ and ε, however, is of mine in the reports of Kgl. Saxon Society, Math. Phys. Class, Vol. XIII, 1861, p. 57 f. and, since the probation has been decidedly successful at collective deviations, it can be considered to be unequivocally valid for such deviations.

5) The probable deviation *w* and quadratic mean deviation *q*. Under probable deviation *w* . of a principal value is to be understood that deviation, which has just so much greater deviations after absolute values about itself, than smaller under itself, thus bez. the deviations Θ ηασ τηε same meaning as the central value *C* bez. the *a*. Under square. Means error *q* I understand briefly the root mean square errors, ie the value that is obtained when the total deviation from a main values *H* particularly raises the squares, the sum of these squares, di$\Sigma\Theta^2$ (probably to be distinguished from the squares of the sum, that of $(\Sigma\Theta)^2$), divided by the total number *m* and τακινγ the root of the quotient, in short

,

Instead of being common for both sides, these values can be just like the. simple mean deviation ε for both sides specially determined and corrected for the finite *m* , to which I do not enter here, by still sparing what is said about it on the supplementary chapter on GAUSS 'law (chapter XVII) to whom these values have definite relations among each other, which permit a derivation of them from each other, which will spare them to be specially performed upon the performance of e among the elements.

6) The extreme values *a of* the table, ie the smallest and largest *a of* the table, the former as *E ', the* latter as *E* , to designate. However, according to the tradition of the table, the higher extreme is at the bottom of the table, and the lower is the uppermost.

§ 10. If two values *a, β* in connected the following way by parentheses, as *a* (*β*) *,* this expression is equally valid with *a β ,* di product of *a* and *β* , but if they are connected by square brackets in the following manner are: *a* [*beta*] *,* so this does not mean that *a* to β should be multiplied, but that *a* function of *β* is; So z. B. Θ [*A*] denotes a deviation of *A,* Θ [*C*] is a value of *C* , *m* [*A*] is the total number of deviations. *A; m* [*C*] with the same bez. *C* usf.

But in the case of the frequent use of the principal values *A* and *D,* since the expressions and formulas relating to them would become inconvenient and unwieldy by such addition, I generally prefer that Θ *, m,* ε βε equally different according to their dependence on *A* or *D.* To put simple names, this is done by the following, under the main values concerned designations, which refer without distinction to the mutual deviations, but after they belong to the positive or negative side special, even with a dash above or below to be provided are:

	A	*D*
Θ	Δ	∂

m	μ	$\boldsymbol{m}$
ε	H	e

So z. For example, Δ ισ a deviation from Δ *,* ∂ ισ one of *D.* Since the total number of deviations is independent of the choice of the principal value, it is generally $m = \mu = \boldsymbol{m}$, whereas ∑Δ ισ not equal to ∂∂ , and η ισ not equal to *e.* is.

The difference $\mu' - \mu_,$ (relative to *A* valid) is briefly denoted by *u* , the difference $\boldsymbol{m}' - \mathbf{m}_,$ (to *D*) by *u* . From *u* follows μ' and $\mu_,$ from *u* follows $\boldsymbol{m}'$ and $\boldsymbol{m}_,$ according to the following equations:

.

,

For the deviations of the upper and lower extremities from the arithmetic mean on absolute values, which can be taken into account several times, the designations serve:

$$U' = E' - A \text{ and } U_, = A - E_, .$$

Instead of considering the total number of deviations, either on both sides or on each side in particular, we will also find occasion to do so, from the principal values only up to certain limits or between given limits, be it their absolute value or their condition to m , m' or $m_,$ after, which is especially discussed using the signs Φ and φ later (in v. chap.).

In the usual way, in the plates, from the small measures *a* to the larger, that is to say, the natural position of the sheet, has progressed before the eyes from the upper to the lower part of the table, which conflicts, of course, with lower values than lower ones , lower; greater than higher, upper values. It is therefore necessary to decide according to the connection or explicit statement whether the expressions "higher", "lower"; "upper", "lower values" are related to the position of the board or the size ratio of the values. To avoid this somewhat annoying formal conflict, it would be better in the future, the distribution boards with the largest values *a*to start; but after following the usual set-up through the previous major part of my research, I could not change it without rebuilding my boards and running the risk of confusing myself. In any case, the bars at the top and bottom of the values refer to the size ratio of the values, not their location in the table.

Afterwards, the meaning and terminology of the following expressions are to be discussed, which play an essential role in our investigations.

By "Vorzahl", "Vorums" I mean respectively the number $\sum z$ and Sum $\sum a$ *of* the *a,* which precede a given value *a of* the table in size, under Nachzahl, Nachsumme which follow a given value *a of* the Tafel in size. Of course, these numbers and sums change with the values *a of* the table which they precede and follow, and in order to prevent expansiveness, I also cite particular names here for the cases which are to be considered in the applications. Generally like with v , V , n, $N$the Vorzahl, Vorsumme, Nachzahl, Nachsumme respect to any eligible start a and closing a be a given distribution panel designated under $\boldsymbol{v}$, $\boldsymbol{V}$, $\boldsymbol{n}$, $\boldsymbol{N}$ the respective values with respect to the a , to which the largest z belongs, the di empirically denominated value D , among $\boldsymbol{v}_i$, $\boldsymbol{V}_i$, $\boldsymbol{n}_i$, $\mathbf{N}_i$, with respect to an *a,*by the way, in most cases it coincides with the previous one, the densest value, where then the designation can also be omitted by the index.

§ 11. Finally, the following remark. It will be an occasion for an arithmetic and a logarithmic treatment of the K.-G. the former being used for such objects whose average deviations in their principal values are only small, the other for those in which they are comparatively large. The first is not only the case, which is far more frequent and therefore more extensive than the second one to be considered, but also easier to handle, and all the provisions and titles of this chapter are to be referred to this case first; but without consideration of the second case of the whole investigation, the necessary universality would be lacking.

The main difference between the two methods of treatment is this:
In the arithmetic treatment, the deviations of the individual are a of their main values in the ordinary sense as arithmetic, di as positive and negative differences taken from their core values and the core values even immediately after specified rules from the a of Distribution panel determined. In the logarithmic treatment, the deviations with which one operates are taken as logarithmic, ie, as differences of the logarithms of a from so-called logarithmic principal values, ie chief values, which according to the very same rules are log a , as the arithmetic chief values from the simple ones $a$be derived. The transition from arithmetic to logarithmic treatment brings with it many new points of view, provisions and designations, which will be discussed later, however, after the occasion has been made to refer to them (see, in particular, Chapters V (§36) and XXI). ,

Under π τηε usual LUDOLF number = 3.1415927, below e the basic number of natural logarithms = 2.7182818, below Mod. = Log. comm. e the so-called modulus of the common logarithmic system = 0.4342945 understood; from which, because of the frequent use of it, it may be useful to cite the common logarithms. One has:

$$\log \pi = 0.4971499; \log e = 0.4342945; \log \text{mod.} = 0.6377843 - 1.$$

The following values are listed under t , t' , t , respectively:

Roger that. Below t- table is a table in the appendix, § 183, which gives the values Φ in relation to t , to be discussed in Chapter V, in the sense of GAUSS 'law of accidental deviations. Since the value exp [- t 2] 7) is of frequent use and somewhat complicated calculation, the calculation of its logarithm may be given here, from which it itself is directly derivable.

7) [For the sake of simplicity, here and below the exponential function *ex* is denoted by exp [x], whereupon exp [- t^2] is *substituted for* e^{-t^2} .]

To find log exp [- t^2] = log 1: exp [t^2], add 2 log t to 0.63778 - 1 (ie to log Mod.), Look for the number in the logarithmic tables and take it negative, Thus you have in it the required logarithm 8) , but in a form that deviates from the usual one and that is itself unsuitable for the application of the logarithmic tables to the derivation of exp [- t^2]. To obtain it in its usable form, subtract its absolute value from the integer higher by 1, and add it to the difference at the back with the - sign. So, if log exp [-t^2] = - 0.25 or - 1.25 or - 2.25 would be found, one would have to set resp. 0.75 - 1; or 0.75-2 or 0.75-3 usf

8) In fact, the logarithm of exp [t^2] is equal to t^2 log e , hence the log. of l: exp [t^2] equals the negative logarithm of exp [t^2].

Under ***E*** the unit is meant in which the copy sizes *a,* the main values *H* and deviation amounts are expressed thereof.

Instead of probability is usually W **.** ; instead of collective object, as already noted, K.-G. and instead of Gauss's law, GG is set for future comment.

III. Preliminary review of the study material and more general comments.

§ 12. An important difficulty for an investigation such as the present one lies in the procurement of the necessary material. Indeed, such can only be found in a plurality of K.-G. from different fields, each of which is in such a large number of specimens, that contingencies of distribution by measure and number are close-for absolutely impossible-can be considered balanced by the law of large numbers, and in each of them the subsequent props can not be regarded as being fulfilled. Finally, the information must contain all data necessary for processing.

But some kinds of K.-G. which could not be passed over to give the investigation the necessary universality, have been nothing at all up to now, and if there is no lack of information for others, yes, for some, such as the measures of the recruits, an *embarras de richesse* is present, *since* not all the demands made on it for the purposes of the investigation are sufficient with them in their current version. However, only a few items are available for one's own measurements, and since it has to be measured and distributed in every very large number of copies, time and patience easily find their limits in this, equally lengthy and protracted business.

In the meantime I have succeeded in bringing together the following material for our investigation, in some cases laborious and cumbersome, of which, of course, some of the requisites to be asserted are incomplete, but there is also the opportunity to show the success of this.

I. Anthropological.

A. Measures of recalculation par excellence, the measures of age of recruits of a certain origin, chiefly Saxon, from whom I was able to obtain copies of the original lists in order to obtain distribution tables in a form suitable for examination. Most important for our general study in the first part are 20 years of Leipzig student crèche measurements with a total *m* = 2047; Soon 17 volumes of so-called Leipzig city measures, ie in terms of recruits of the rest of Leipzig's population, with a total -*m* = 8402; also recruits of 3 years, resp. the Borna'schen and Annaberger Amtshauptmannschaft with *m* =2642 and 3067. For this purpose, in the second part recruiting matrices rel. other countries, as far as such proposals are concerned and have been dealt with earlier by QUETELET, in particular Belgian, French, Italian and American, a partly critical discussion, partly by the deviant treatment of Quetzel; and measures of body weight and chest circumference of the recruits to be taken into account.

B . **Skull dimensions** , which have been proposed to me by Prof. WELKER in Halle, a) the vertical circumference, b) the horizontal circumference of 450 European men's skulls.

C. Weight of internal organs of the human body , according to BODY's statements [1)].

[1)] [Dr. Boyd's Tables of the Human Body and Internal Organs. Philosophical Transactions of the Royal Society of London; 1861.]

II. Botanical.

Rye ears (*Secale cereale*) of the same location and age, measured by myself , 217 six-membered (except for the fruit ear) and 138 five-membered; each of the members especially measured and partly as a special K.-G. treated, partly taken into account by its relation to the other members.

III. Meteorological.

a) Thermal and barometric daily and monthly values or deviations in the sense to be discussed in § 19 and 20 in more detail. These include those of QUETELET in his *Lettres sur la prob* . listed, under § 21 to be discussed, 10-year-old so-called " *variations diurnes* " with a m of 282 to 310; For this purpose, our own compilations of thermal and barometric daily values after observations on the Peissenberge over a longer series of years, and of thermal monthly deviations according to DOVE's treatises.

b) Daily heights of fallen water for Geneva through many years, compiled by the Bibliothèque universelle de Genève (Archives of the sciences physiques et naturelles).

IV. Artistic.

a) Business cards and address cards of merchants and manufacturers, especially measured by myself in length and width.

b) dimensions, height h and width b , of gallery paintings (in the light of the frame) to the catalogs of the collections with reduction to the same unit of measurement for genre paintings, landscapes, still lifes especially determined by me; The case is differentiated where $b > h$ and where $h > b$.

This only for a preliminary overview; More specifically, the above material will be dealt with in particular chapters of the second part, where the more detailed information to be found here will be found, and also referred to, if reference is already made to this material in the first part.

It may be remarked that among objects of the past there are those with which there is little or no interest in the subject. But the point of factual interest in it has not been at all decisive here for their choice and treatment; but only their usability as a basis for our investigation, in which respect some seemingly insignificant objects, such as the dimensions of the gallery paintings and the daily rain heights have become important.

But insofar as there was an objective interest in the objects, one must not, for the same reason, expect to find their treatment exhausted in this interest, even if many of the results which enter into it will automatically decline as by-products of treatment. Each of these objects could give rise to a monographic treatment; but a work as large as one would require only the measures of the recruits, should a comparative presentation and discussion of them be carried out for the different countries and in the same countries for the different vintages, or for the cranial dimensions of the different races, or for the structure of the different Gramineae! At bushings of this kind is not to think here. On the other hand, that makes[2)]

[2)] [Note: It should be added to the information in this chapter that a partial replacement of the specimens was necessary since, apart from fractions of the size of the recruits and the dimensions of the rye straws, none of the designated K.-G. Url lists or primary distribution panels were found. To be sure, as far as practicable, the research material was supplemented from the given sources; In particular, dimensions for gallery paintings were added to the catalogs of the old Pinakothek in Munich and the Gemäldegallerie zu Darmstadt; for the daily rain heights of Geneva the Archives of the sciences physiques et naturelles the bibliothèque universelle taken (see chapter XXI, as well as XXVI and XXVII). But instead of the observations of thermal and barometric daily values on the Peissenberge, corresponding values were published for Utrecht in the Dutch Yearbook of Meteorology (see Chapters XXIII and XXVII). Finally, the replacement for the skull dimensions (see Chapters VII and XXII) I owe to Professor WELCKER, who was good enough to give me the measurements of about 500 European male skulls.]

IV. Props; Abnormalities.

§ 13. Should a K.-G. To permit a successful investigation, he must fulfill certain conditions, some of which are in his conception, and in part subordinate to more general points of view.

According to the introductory statement, a K.-G. be an object of indeterminate number which can be grasped under a certain concept and randomly fluctuated in its quantitative determinations. Now there can not be an infinite number of copies of it, but it is necessary, as has been said, to obtain as many as possible from him, so many that the strictly taken, ideal laws of chance, which can only be claimed for an infinite number, still have one for the desired degree Accuracy of adequate approximation can be confirmed. But if this condition is sufficiently fulfilled, a K.-G. nor be normal or flawless from other points of view, as we may like to briefly express, to comply with the legal provisions which are considered the most general for K.-G. let set up,

This includes above all that the specimens from no other point of view to a K.-G. taken together, nor are any of them excluded, as being grounded in the concept of the object, that is, that the object is not only multitudinous from the previous viewpoint, but also in proportion insofar as all the specimens which it presents within the limits of its concept are actually counted It is not because of this or that secondary consideration that one or the other part of the scale of measurement comes to an end, that herewith the object is mutilated so to speak, as it is, for This would be the case, for example, if the so-called subordinates were to be excluded from recruiting matrices, whereas, on the other hand, the object must also be kept as pure and unmixed as possible, ie specimens. who, according to any one side, should step out of his concept, be excluded from him; For example, where the collective term refers to healthy individuals, specimens with pathologically altered dimensions must be eliminated; Therefore, neither in the WELCKER skull measurements to be treated by me, neither barrel-shaped hydrocephalus nor decidedly enter into microcephalic skulls. But this is followed by comments of general significance.

§ 14. It is certain that the boundary between healthy and abnormally altered skulls can not be determined with certainty, and a corresponding uncertainty about the delimitation of the object returns in many other cases; but if only the uncertainty keeps itself within such narrow limits that the limits of uncertainty, which one must submit to because of unbalanced contingencies, are not exceeded, then no considerable disadvantage can arise on the whole, and one becomes one through success itself satisfied if the object delimited at best disregards the normal distribution laws, or if one can cut off so many copies that it does.

However, this raises the following very important question: It is of course logically self-evident that if healthy individuals or parts of such, such as cranium, are to be examined with regard to the distributional relationships of their specimens, those who are recognized as ill or who have been accepted are not included and no less self-evident that the determination of the conditions for healthy specimens has a greater interest than for a mixture of the healthy and the ill; only it seems contrary to the generality of the task of the collective gauge, to determine the most general distribution laws the K.-G. from mere healthy specimens to the object of a mixture of the healthy and the ill.

In fact, when the abnormally altered skulls emerge from the concept of the healthy, they still fall under the concept of the skull in general, and what justifies us in seeking the most general laws for K.-G. to dispose of the diseased cranium, since, on the contrary, we would have to apply only the broader concept, which includes all the skulls, instead of the narrower one of the healthy; and there are countless other cases where there is an equal possibility of the narrower and wider version; strictly speaking, such exists everywhere, since at last all K.-G. can be united under the concept of an existing being, which can only be narrowed down in various directions. However, we would be tempted to try our generally published laws on very broad versions of the K.-G. to prove, to drive poorly, if they did not prove themselves or only partially, but in sufficiently narrow versions for the most diverse K.-G. remain the same and thus prove their universality. Now, one wonders which viewpoint is decisive for the restriction of the distance to be observed.

This seemingly difficult question must be answered with regard to the following actual circumstances.

which are unanimous, and which are composed of disparate objects. Any extension of the term of a K.-G. but carries with it a compound of one or more other, possibly disparate objects.

From this point of view, it is immediately obvious to many objects that they must not be mixed. In fact, nobody will think of it, men and women or children and adults in the same K.-G. when the distribution of their specimens is to be considered in terms of body length, even though they are collectively covered by the broader concept of human beings; but one knows in advance that there are essentially different averages for making them disparate objects. And so must a composition of healthy skulls with pathologically altered skulls to a K.-G. be found inadmissible in so far as both behave disparately against each other.

§ 15. From this point of view the results of investigations on the measures of the recruits seem very instructive, which, having been briefly mentioned above (chapter III under I. A), are to be communicated in more detail in the second part of this work (chapter XXIV) ,

In general, recruiting measures can be grouped together for the most diverse countries, times, ages, under the broadest terms of such measures, but also very specialized; and from the beginning you will z. B. 18 year-old recruits of one country do not want to be mixed with 20-year-olds from another country, as both differ in different median sizes; but also recruits of the same country of the same age permit specializations in different senses. For example, I treated the recruits of (2 year old) Leipzig students on the one hand, and those of the rest of Leipzig, the so-called Leipziger Stadtmaße, on the other. For the first, there has been a very satisfactory confirmation of the general distribution laws to be drawn up, for others, according to some relation, imperfect confirmation. which I call fundamental, yield; in comparison with calculation and observation, it has been shown that in the case of the latter the small measures occur relatively more frequently than they should have been calculated on the basis of the fundamental laws, without unbalanced contingencies sufficing to explain them. The same was true for the recruiting measures of the mixed population of various larger districts of Saxony. What is the difference between the first and the other cases? The recruits of the students refer to the limited extent of relatively wealthy estates that do not fail normal growth of individuals; the others on individuals from a mixture of such estates with stalls, in which there is more or less of such resources from conception and birth,

Add to my command 20 years transitions from Leipzig student recruits dimensions with a total m = 2047, only a single individual drops (60 inches) below the level 64 inches [1)]; in seventeen vintages of the size of the rest of the population of Leipzig (Leipziger Stadtmaße for short) with a total m = 8402, 197 individuals fall below 64 inches (the smallest at 48 inches); and we reduce 197 by the ratio of the total m,For instance, against one individual of the Leipzig student masses, 48 of the Leipzig city measures still fall below 64 inches. But the mixed population of Leipzig, like that of every great city, contains a large percentage of the miserable proletariat. But further: 3-year recruits of Borna's local authority except Leipzig (preferably including small towns and farming villages) with *m* = 2642 gave absolutely 50 or, as previously reduced, 39 measures under 64 inches (with the minimum measures 51 inches), and 3 vintages recruits the Annaberg County Commission (including many mountainous and poor factory populations) with *m* = 3,067 absolutely 62, reduced 41 measures below 64 inches (with the minimum dimension 49 inches). So according to the proportion of *m* we have at all relevant for the specified 4 departments:

1 48 39 41

Measures under 64 [2]) , and if we go over to the arithmetic means (after the primary tables), the following values are found in Saxon customs:

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71.76 69.61 69.34 69.00.

Thus the arithmetic mean of the Leipzig students is more than two inches larger than that of the mixed Saxon population, and the same applies to the central value and the densest value. On the other hand, the mean deviation with respect to the arithmetic mean is, according to a uniform method of determination for all departments, in Saxon customs duties for:

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2.01 2.26 2.14 2.33.

And, of course, the difference between the two relations would be even greater if the mixed population of the last three divisions were divided into those with normal and those with abnormal growth, and both could be contrasted.

1) [1 Saxon inch = 23.6 mm.]

2) Less noticeable than the smallest measures, the difference between the student dimensions and dimensions of the other three sections is the largest; and the distributional calculation of the latter is better than downward; but a difference in the largest dimensions is not entirely missing. The student measurements closed up with the three measures 80; 80.75; 82.5; the Leipzig city measures with 79.5 (4 times) and 79.75; the Borna people with 77.25; 77.75; 78.25; Annaberg's with 76.75; 77.25; 78.5.

It can not be asserted that if we had the proletariat recruits for ourselves as well as the wealthy classes in the students, our fundamental laws of distribution would be as valid in those as in them, because the proletariat itself is still one far concept is, which of the specialization is capable in different directions, and not *apriori*It must be assured that his specialties are unanimous in the above sense. In the first place, the same would be true of the wealthy classes represented by the students; but as experience itself teaches that the specialization in student masses is sufficiently advanced to permit the affirmation of the laws in question, as far as it is possible at all for unbalanced contingencies, we may at the same time calm ourselves, whereas we here and there to have the specialization even further if it was not enough.

It can also be admitted quite well that if we only increased the *m of* the degree of the student's crèche, and then from different points of view, e. B. secreted depending on the origin of villages or towns or from different years or different stands in departments that still, sufficient *m* would have to be able to discover subtle differences of the essential elements for sure, it would be no lack of those which a run counter to complete unanimity; and it does not prevent anything from making a task of inquiry from it.

But if these differences are only small, and the many divisions which can be made in different ways, herewith vary the differences between the elements themselves and the character of chance, not only can reasonably be presupposed, but the fact itself teaches that the respective differences of the elements in the unavoidable unbalanced contingencies are indistinguishable, and that the verification of the fundamental laws is not a major obstacle.

§ 16. But the less allowed one may be in the deviations, which are the distributional relations of widely divided and thus ambiguous K.-G. From the fundamental laws, we see a contradiction to these laws, as it in principle suffices to know the relations of mixing and essential elements of the composing objects of an ambiguous object, and to compute the distributional relations of the compound object according to the fundamental laws themselves; to assert its general validity also in this respect.

In general, it follows from the foregoing that, in ascertaining and examining the most fundamental laws of distribution, we must not only guard against the distributional results of widely distributed, indiscriminately mixed objects, contrary to widely divergent directions, against the universality of the laws employed for sufficiently narrow, unified subjects but also in the choice between the results of a wider and narrower version, under otherwise similar circumstances, which are preferable to the narrower ones for the establishment of the fundamental laws. The previous considerations are essentially subordinated to the following.

The origin of the copies of a K.-G. from different spaces or times or both at once leads not only to qualitative but also to quantitative differences of the same, which deserves special attention insofar as, in order to obtain a sufficiently large *m*to obtain for a successful investigation, usually causes or coerced, the K.-G. they can not belong at all to compositions made up of specimens which belong to different spaces or times, indeed to the same space and time. In this relationship, a conflict now takes place. Bringing the specimens from very remote or very wide spaces and times places them in danger of uniting disparate objects and thus of missing the fundamental distributional relationships; Gathering the specimens from space and time limits that are too narrow gives great scope to unbalanced contingencies in order to deduce essential provisions with any certainty whatsoever. However, the limits to be respected in this regard can not be drawn a priori, and, finally, success itself must decide whether the assumed temporal or spatial breadth of the object leads to a satisfactory fulfillment of the fundamental laws of distribution; where not, the narrowing continue to drive, and if you do so in too small values of*In* order to obtain results of sufficient certainty, the investigation is abandoned until a larger number of specimens are obtained. In general, this is probably the most practical.

§ 17. In the question of whether an object is composed of disparate components, particular attention must be given to the following, in part, already touched relations of the distribution tables.

It is well founded in our fundamental laws that the *z increase* continuously with the *a* up to a certain size of the *a* , but with continual growing *a* likewise decrease continuously, so that there is a maximum of the *z* in a middle part of the distribution table (at the so-called densest values) and two minima respectively at the beginning and the end of the table (at the extreme *a*). If one takes the *a* as an abscissa, the *z* as the ordinates, one can thereby graphically represent the legal distribution in a known manner and thus obtain a curve which, in the case of small *i*rises smoothly to a summit and descends from there. But in the so-called primary plates, that is, directly derived from the original lists of measures, one will generally find from the beginning through the whole plate an irregular rise and fall of the *z* with continuous growth of the *a* , hence a hunched quality of the distribution curve; to which the primary distribution tables of the seventh chapter give sufficient examples. The most general, yes, never missing cause of such irregularities lies in any case in unbalanced contingencies, and the dependent on this cusps of the curve disappear by a sufficiently far driven reduction of the blackboard, ie according to earlier (§ 6) stated explanation, take the *z*for equal intervals of *a* through the whole table, as in Chapter VIII, and to give examples of reduced tables. But in part, the cause may lie in the fact that K.-G. of disparate nature of their home values.

In fact, from a general point of view, it can be overlooked that, for As did the dimensions of the same amount of men and women who are very different in the arithmetic mean as densely worth and mixed in, so significantly, that is, apart from unbalanced accidents, a rise to the emergence of two maximum *z* therefore two closest values would arise indeed, by mixing even more disparate objects, distribution boards with much more maximum *z could* be created. In any case, only distribution tables with a maximum *z* are suitable for testing the fundamental laws of distribution in the main panel of the panel, whereas small irregularities towards the ends of the panel are without significant disturbance. If, therefore, there are distribution tables which do not correspond to this condition, they are only useful for the consideration of the laws after such reduction, that they correspond to them by sufficient equalization of the contingencies, according to which the laws in question can be very well confirmed on the reduced table, if the majority of the maximum *z* really depended in the main Bestande the board only by unbalanced contingencies.

But is not to be overlooked that, as can be by reduction of a distribution panel whose intervals increased, at the same time, dependent with the unbalanced coincidences of disparate nature of the components of the panel, the majority of the maximum *z* may disappear if this namely on each other near *a* , which together enter into the interval increased by the reduction, become indistinguishable, indeed, one only has to go as far as possible with the reduction and thus increase in the intervals in order to achieve this safely. Thus, although the rule, the panel to be tested with respect to the distribution, is reduced by reduction to only a maximum z and from there to both sides descending aisle of the $z$but any deviation from the fundamental laws may still depend on a disparate nature of the components of the tablet which has been blurred by the reduction; Consequently, even in this respect only the study of the distribution itself can be decisive.

§ 18. However, we are not finished with our props yet. Objects designed by man with regard to certain purposes or ideas, in short we call them artistic, are subject to collective law despite the intention which has been obscured in their creation, but with regard to determinations of size which still leave chance to chance; but if secondary considerations or secondary purposes essentially limit the freedom of chance by favoring or excluding individual dimensions, then the laws can also be essentially aborted, as illustrated by the following examples.

Business cards, as well as the so-called address cards of merchants and manufacturers, are varied in the most varied manner according to length and breadth, and I thought at first to have an excellent object for examining our laws, since they were in large numbers, be it everyday Traffic, be it from the pattern books of their makers, in which specimens are glued (of which I have used many of different manufacturers for measurements), while giving the advantage that the accuracy of the measurement and estimation more than many other objects in the hand. But though they are by no means wholly removed from our laws, whether by length or breadth, they are but a very imperfect proof of them.

In spite of the variation in their dimensions, the freedom of chance is limited by the fact that the fabricators generally prefer dimensions which make it possible to make the most of the cardboard sheets from which the cards are cut, ie to consume them as completely as possible , particularly popular ratios between latitude and longitude, especially 2 : 3 or 3: 5 (approaches to the golden section) to comply; and indeed, in the measurements of such maps, which I have made in the sample books of a majority of manufacturers, I have convinced myself that in each of them certain dimensions occur more frequently than could be considered accidental. The dimensions of the gallery paintings in the light of the frame, however, are not subject to the same disadvantage, and, having brought together a large quantity of them from the catalogs of the most varied galleries (see Chapter XXVI), will furnish an excellent material for the proof of the logarithmic laws of measure.

§ 19. In the case of the natural objects, on the other hand, one of the requisites conditioned by the concept itself is that the specimens do not stand in a natural legal dependence on one another, which emerges from the laws of chance. This point comes especially by meteorological K.-G. in consideration. Thermometer and barometer readings, as well as other meteorological values, show in every place a legal ascension and disassembly, disturbed by contingencies but resolutely in mean values, already in the course of the hours of a day, not less by the days or months of one year. These so-called periodic meteorological values do not fall under the concept of a K.-G., but only the non-periodic ones inasmuch as they can be considered as randomly changing. In this regard, we can shortly provide meteorological daily values, monthly values and annual values, insofar as they deviate from their means of many years, and these deviations themselves as daily deviations; Monthly deviations and annual deviations differ, something which will be more specific here, as there will often be occasion. to come back to such. We tie the explanation to the thermal values and deviations, which results in the transfer to other types of meteorological values and deviations by itself. to come back to such. We tie the explanation to the thermal values and deviations, which results in the transfer to other types of meteorological values and deviations by itself. to come back to such. We tie the explanation to the thermal values and deviations, which results in the transfer to other types of meteorological values and deviations by itself.

Thermal daily values, in particular, can give each person particular day according to his annual date, say z. For example, the 1st of January. Let us take as the temperature of this day at a given place in a given year, for a short time the thermal value of the 1st of January, be the average of its 24 hours, or the temperature of a certain hour of the day, or even the mean of the maximum and minimum temperature of the day. This daily value of January 1 has been observed for a number of years behind each other. The randomly changing daily values after the years represent the copies *a*of a temporal K.- G. From this, we take the arithmetic mean by dividing the sum of the daily values by the number of the same, which coincides with the number of years through which we have observed. This means the overall thermal hot daily average of the 1st of January, and the deviations of the daily values obtained in different years *a* of the general daily average *A* then form the individual daily variations, which according to the above notation with Δ are to be designated. Such provisions may be obtained in particular for January 2 and every other anniversary at each site.

However, instead of for each day of the year, such provisions may also be obtained for each particular week of the year, for each month of the year and for the whole year itself from multi-annual observations, which then include weekly, weekly, monthly, monthly, annual, annual variations are denote. Of these, the monthly thermal values and monthly deviations deserve particular attention because of the numerous provisions in many places. The thermal monthly values as *a* are thus obtained z. For example, for January (and correspondingly for every other month) in the mean temperatures of January, determined by a series of years, which are to be obtained from the 31 days thereof; the thermal monthly deviations of January as Δ in the deviations of these *a* from the general mean of *a.* Instead of arithmetical mean and deviations from it, other main values and deviations from them can be derived from such values.

Meteorological K.-G. of this kind are at all estimable for the investigation of their general laws from several points of view; secondly, because of the abundant material available in or from the sources of meteorology; secondly, because of the accuracy of the determinations made by meteorological observatories and methods; and thirdly, because these objects are the only material to judge by , whether temporal K.-G. subject to the same laws as spatial. Only they suffer from the very important disadvantage is that because the *m* same coincides with the number of years by which rich observations, not easily a large *m*the same, indeed nowhere, has existed so far as would be desirable for the safety of the results to be derived therefrom. [3)]

[3)] Among the 70 places for which dove notes the thermal monthly deviations in one of his essays, it is merely Berlin, where 100 is exceeded as *m* , by passing through 138 years, and only Prague and London show a *m* over 90, respectively 94 and 92.

§ 20. Now, however, one can obtain a much larger *m* from a given number of years, than the number of years, in the following way, which, in the case of important doubts, can not simply be rejected.

To start from the definite notions of a QUETELET example (see quete-let's lettres, last vertical column of Table p. 78), we assume that the temperature of all January days is the mean between the minimum and maximum temperature of each day at a given time Places (Brussels) has been observed through 10 years, then we will according to the specified method of determination, which is to be regarded as correct, receive for each of the 31 January days as K.-G., the first, second, third, etc., a *m* = 10, which is too little to study the distribution laws; against this we will be a *m*= 310 for the whole January month as K.-G. If, after quetelet's procedure in the example in question, we take the 31 daytime temperatures of January as copies of the January daytime temperature for the 10 years, give 310 copies, from which the arithmetic mean by division with 310 draws, of these the 310 Take deviations Δ and, if we wish, also determine the other principal values with the deviations from them.

Of course, it is clear from the very outset that, apart from the accidental changes, the temperature of January increases legally from the first to the thirty-first day, we hereby obtain a complication of accidental gait with a natural-law course of daily values, but strictly the natural-law Gang should be excluded when investigating the essential distribution laws. However, it may well be admitted that the changes in the temperature of the day, which are caused by the legal progress of the same during one month, are too little considered in comparison with the average size of the accidental changes of the individual daytime temperatures, in order to disturb the laws of chance considerably; in any case, they can not cancel the same but just disturb it. But a more important concern arises that quite apart from the legal progress of one month, the meteorological conditions of the immediately following days everywhere betray a certain dependence on each other, which is not provided for in the laws of chance. In general, several warm, one above the middle of the value of the temperature of January, and several cold ones, the days falling below, follow each other, and the transition from one to the other does not occur by leaps and bounds, but by successive ascents to one certain height above the middle of the value and, since the rise can not go into the indefinite, re-sinking to a lower height or below the middle of the value, except that no regular periodicity is visible in this change between ascending and descending. Similar to all so-called

To this end it seems only useful to remark that there is a very simple way of convincing oneself of the demands of pure chance for such cases as the non-satisfaction of these cases. For a number of years, I have obtained the draw lists of Saxon lotteries in which the winning numbers are listed in the order in which they came out. If anywhere, chance plays its role here. If we denote the even-numbered numbers with a +, the odd-numbered ones with a -, and trace the series of characters through a large number of consecutive numbers, we find, apart from a small difference due to unbalanced contingencies, just as many sequences of the same characters as a change of unequal. If, however, we do likewise with the + cases and - cases below the value center determined from the totality of cases in meteorological daily tables, then the number of consequences outweighs the change, proof of a dependence of the consecutive meteorological daily values emerging from the random laws. Further, if, instead of the previous denomination of the consecutive lottery numbers, we denote each overcoming of a number by the following with +, each descent of the following among the previous ones, we find in pursuit of a large number of numbers (apart from unbalanced contingencies) the Number of bills twice as large as that of the consequences; but we do so with a corresponding designation of the consecutive meteorological daily values, Thus, the number of changes lags far behind the double number of consequences, second proof that the rise and fall of the meteorological values from day to day does not obey the pure random laws. One complements and intensifies this investigation, which I now only hint at, in order to return to it in a later chapter, in that, in addition to the deviations from those laws of pure chance, which strictly for infinite*In* addition to the fact that *m* equivalences are to be taken into account by unbalanced coincidences, so too does the probabilistic and mean deviations from the statement of the laws dependent on the finiteness of the *m* , for which in fact formulas can be established.

From an in-depth investigation has now revealed to me [4]that, while the meteorological values of successive days of the same month show the given characteristics of dependency to an eminent degree, even the monthly deviations of successive years are not entirely withdrawn, even if they show so weakly and little decidedly, in order not to be considerable in their use To be allowed to obtain disturbance of the laws of chance; and this object undoubtedly deserves an even more extensive and extensive investigation on the part of professional meteorologists with the help of those criteria in the interest of meteorology itself, as I have allowed it here to be part of it, where it was only in the interest to determine which K.- G. are at all suitable for the examination and application of the pure laws of chance.

[4] [In XXIII. Cape. Evidence given.]

Meanwhile, important to note that the excluded translucent on the previous option, the random laws on meteorological values showing a dependence of the type mentioned by each other to apply, could be restored in the event that at very large m the dependence conditions change even randomly ,

For illustration, let us imagine an urn with infinitely many white and black spheres, marked with numbers corresponding to the quantities of deviation from a given principal value, such that the number of occurrences of each of these kinds of spheres is equal to the number of Occurrence of the corresponding deviation values as they exist for pure random laws. Thus, in the case of symmetric probability, the law of GAUSS concerning deviations from the arithmetic mean, and in case of asymmetrical probability, our general law to be discussed later, is thus represented; whereby white spheres show positive deviations and black spheres negative deviations. Now happens quite a lot of trains randomly from this urn, In this way the drawn bullets, in their relations, will properly represent the law in question, apart from the unbalanced contingencies left over by the ever-finite number of puffs. But the same will be the case when two, three, or more spheres, which are close to each other in their values, whether according to a certain rule or not, are glued together, so that they can only be extracted together; only a larger number of trains, a larger one glued together, so that they can only pull out together; only a larger number of trains, a larger one glued together, so that they can only pull out together; only a larger number of trains, a larger one$m$, in order to obtain an equally good satisfaction of the laws in question, as is the case with loose bullets.

Of course, the question of whether it is analogous to the meteorological daily values can not be considered settled by this analogy, which merely shows that it might possibly behave that way. But not only is Quetelet's example (Lettres p., 78), with $m = 310$ (in reality, but rather due to the absence of an observation day), closely examined by the distribution of its $z$quite well, but also by thermal and barometric examples with far greater $m$which I myself examined (see chapter XXVII) speak for the same, so that it can at least with the greatest probability be validated, which may be of interest not only to our teaching but also to meteorology. QUETELET himself did not respond to the question.

§21. Incidentally, it is highly desirable that a meteorological example should be available in which the occurrence of numerous individual cases is combined with a lack of dependence of the successive cases on each other. In the Bibliothèque universelle de Genève (archives of the sciences physiques et naturelles) is found in each Monatshefte a meteorological table for Geneva [5)]in which among other columns, which are valid for thermometers, barometers, etc, also a column with the headline; "Eau tombée dans les 24 heures" is given, which indicates the amount of fallen water in millimeters for each rainy day of the month in question. Now, however, several wet and dry days follow each other, but - and that is what matters to us, and of which the analogue is not the case with the consecutive thermal or barometric daily values, - the rain heights collected in the rain gauge following each other Days do not betray size dependence on each other. In fact, even at the most superficial glance, the rain heights of the column in question can be seen to change in the most irregular manner, and not infrequently to follow the tremendous level of rain one day, a very low the next day, or vice versa. But decisive in this respect are our above two criteria; and it is noteworthy what other results they give in relation to the daily rains of rain, as understood in previous senses, than to the thermal and barometric values of the day, for which later (chapter XXIII) evidence will be found.

[5] Another, correspondingly furnished table for the meteorological station on St. Bernhard.

Accordingly, I have not bothered to take the data contained in the Geneva journal for the Geneva rains of all the vintages through which they reach, and after the 12 months I have formed 12 divisions, each of them having a special treatment .-G. represents. In it are z. For example, as examples *a* of January, not only all the rain heights (indisputably mostly from molten snow) that occurred in a month of January, but taken together in the January months of all the years through which the rains have followed, and thereby becomes get a very substantial *m* every month . Of course, it could be arranged that this effort was in vain for our purpose, because it was not*a priori*to assert that the rains are in general subject to the same laws of distribution as the dimensions of the recumbent, the dimensions of the skull, and the like. etc .; but, on the contrary, it has paid off by the fact that the heights of rain with the dimensions of the gallery paintings have hitherto provided the only material on which our logarithmic law of distribution can be proved by striking with proportionally a tremendous asymmetry which makes the principal values far apart offer very strong mean deviations from the main values, thus avoiding the applicability of the arithmetical treatment (see chapter XXI, as well as XXVI and XXVII). And it is undoubtedly his particular interest that such different things as the dimensions of the painting and the heights of the rain should be so determined and peculiar laws of distribution as we will have to set up,

By the way, there is another case of meteorological daily values of corresponding succession independence, to use this short term, as the daily rain heights show, which is all the more necessary to go into more detail than is included in the empirical evidence of our study and Of Quetelet himself to his own in a manner which, in my opinion, is certainly not valid, in which respect I shall return to it several times. These are the so-called *variations diurnes*of QUETELET, of which QUETELET in his Lettres p. 174 fg., With tables p. 408 to 411, while I myself am in the Cape. XXVII come closer to it; Here, however, merely the nature of the same provisionally determined and envisaged with respect to the independence in question.

It has been said above that QUETELET has established the temperature of all the days of each month as a mean between maximum and minimum temperature of each day (for Brussels) and has continued this through 10 years. The difference between the two temperatures, whose mean is the daily temperature, is what QUETELET calls " *variation diurne* " (daily variation). It must be remembered that this deviation of the two extremes of the day from each other may be great or small at the same middle temperature between them, that is, the same temperature of the day, and consequently the succession *dependence* , which the daytime temperatures show, is not at all necessary for the *diurnal variations*needs to extend. In fact, the same daytime temperature, z. B. of 10 °, as a mean of 9.5 ° and 10.5 °, from 8 ° and 12 °, from 5 ° and 15 ° emerge, what variations resp. of 1 °, 4 °, 10 °; yes, if the temperature remained constant in one day, it could still be so high or low, and the variation would be zero. As QUETELET has followed the temperature of the days of each month for ten years, which are given as copies of a K.-G. the corresponding *variation diurnes* , in which one can see specimens of another K.-G. Although QUETELET has the *variations diurnes*does not specialize for all days of each month, which would have required tables of tremendous size without giving the possibility of concise summary, but he has p. 410, 411 tables in which it is indicated for each month how often during 10 years the variation diurne was between 0 ° and 1 °, between 1 ° and 2 °, between 2 ° and 3 °, etc., short reduced interval tables in the sense our later (VIII) chapter.

Now, as noted above, if the *variations of* their size appear to be essentially independent of the magnitude of the daytime temperatures between them, and consequently do not necessarily share their succession dependence, such dependency seems to contradict the tables of the monthly *variations diurnes* at a *m,*which varies for the individual months between 282 (February) and 309 to 310 (January and August), show such a regular course and such a good correspondence with the otherwise valid laws of asymmetric distribution, as one would hardly expect with existing succession dependence; meanwhile, that of QUETELET p. 78 given table of daytime temperatures of July compared with the corresponding table of *variations diurnes* p. 411 that the course of the *z* in both tables is similar and equally regular, so that even without accepting the relevant independence, according to the first principle discussed, this table could be considered useful in the sense in which it is done by us.

§ 22. Hereinafter the following general remarks:

In general, I will become points whereby K.-G., even with sufficiently large *m,*that is, apart from unbalanced contingencies, that we may evade the probation of our laws, as improprieties or abnormalities, but objects which are free of them may be considered as free from thievery. The anomalies, as we see, are of various kinds, and may affect the validity of the laws in very different respects and to very different degrees. It can be counted among the general tasks of the collective theory to ascertain the influence of these abnormalities, which can happen partly theoretically with regard to the distribution laws recognized on the faultless objects, partly empirically, and indeed the latter in a twofold way. On the one hand, one can follow the success of the anomalies in the abnormal examples themselves which reality offers; Secondly,

Here is another field of investigation for others, since I have the same thing about the already so complex task, the circumstances of the K.-G. On the assumption that they were flawless, they were by no means sufficiently settled.

In every respect perfectly error-free objects with a large *m* are scarcely to be procured in the multiplicity of possible errors, and it is therefore with the objects empirically used to establish or prove the fundamental laws of the K.-G. apart from the deviations from the ideal legal distribution ratios due to the finite nature of the *m* and the size of the *i*To allow deviations due to lack of fulfillment of the props or, in short, because of defectiveness insofar as they are kept within sufficiently narrow limits, so as not to raise objections against the validity of the established fundamental laws, of which there is always a degree of latitude for the subjective discretion. Terms and conditions that both the deviations due to the finiteness of *m* as due to size of the *i* , as are withdrawn due to lack of compliance with the props, I call hereafter, except the already used printouts fundamental, even normal or ideal, if only in reality occur in approximations.

Incidentally, from the above, in which, in spite of the fact that it can count itself from the points of view given in the foreword to the exact doctrines, the difficulty lies in bringing it to definite results in its applications. There are other points than exist for physiology and psychophysics in this respect; but they have a similar success. After all, it remains a privilege of all these doctrines to be more precise, first of all to impart security as far as possible, secondly to lead to general laws.

§ 23. The previous remarks concerned props which the K.-G. have to fulfill themselves; but there are also props that the investigation has to fulfill. The distribution boards can be set up in more or less expedient or usable form, as described in Chap. VII and VIII is more specific. The inevitable mistakes made in measuring the specimens; must not be insignificant enough to interfere with the enforcement of the laws, and the accuracy of measurement will therefore generally be sufficient to neglect the measurement errors against the collective deviations. In the measurements, the departments indicated on the scale still maintain an estimate by subdivision; and this is very common that the whole and half divisions are favored, which I call the error of uneven estimation, and of which I refer examples. the size of the recruits and skull dimensions in Chap. VII lead. Such errors may be detrimental to the precise determination of the elements, and it is therefore necessary to be on the lookout and, where such exist, to render them as harmless as possible by means of an appropriate reduction. With the amount of measures to be taken, oversight in the measure itself or its recording is all too easily possible, and there may be no other means of avoiding it safely than making the measurements twice independently of each other and controlling them, as I have done done by measuring the rye ears; but since the laborious work is thereby doubled, you will hardly understand it anywhere. It is even more difficult to avoid oversight by utilizing a large amount of measures for determining the elements and proving the laws; and at least with respect to any conspicuous or important result, control by repeating the calculation is not to be avoided.

In general, there are certain and uncertain ways of determining the elements, and of course the first ones are preferable in nature; but since only approximations to the ideal values of the elements are attainable, it may be that a small advantage in this respect does not come into consideration against the relief, which gives a somewhat less sure way, and so from a practical point of view but to be preferable if it is sufficient to state, with satisfactory certainty, what one has in mind. Astronomical accuracy and certainty can not be achieved in this case, and it may be that the futile claim to achieve it makes an investigation impracticable.

V. Gauss's law of random deviations (observation errors) and its generalizations.

§ 24. After GAUSS [1)] not only theorized the Basic Law of so-called observation errors, ie the accidental deviations of means of observation, but also the same has been proved empirically by BESSEL [2)] , it could be assumed that it only applies , this law on the random deviations of the copies *a of* a K.-G. from their arithmetic mean *A,* that is, to the Θ with respect thereto, in order to have the same as for the observation errors, ie to have a law which, after empirical determination of the arithmetic mean and a principal deviation value with respect thereto, like the mean deviation$\varepsilon = \sum\Theta$ *: m to determine* the whole distribution of a K.-G. by measure and number, ie to determine in what proportion to the total number *m*(provided that this is not too small) specimens in any size limits of deviation from Means occur.

[1)] [Theoria motus corporum coelestium, 1809. Lib. II, Sect. III. ó Theoria combinationis observationum erroribus minnimis obnoxiae; Commentation societ. reg. Scient. Götting. rec. Vol. V. 1823.]
[2)] [Fundamenta astronomiae, 1818; Sect. II.]

Since we now have the task, a general distribution law for K.-G. to find out, at least from the GAUSS'schen law (short GG) will go out, repeatedly have to come back to it, and indeed in a certain limitation for K.-G. To find ourselves sufficiently adequate, only to be subordinated to a more general law, there must be something in advance about this law. Although it has long been known and familiar to specialist astronomers and physicists, on the basis of this they calculate the probable error made in the determination of a means of observation; but I have here also to presuppose other circles of readers and other uses of the law and therefore, rather than relying on the unpopular integral term of the law, from the easily comprehensible tabular expressions into which the same can be translated and for which practical application must everywhere be translated anyway. Later (chapter XVII) will be returned to the same at the end of its integral term; for now the following will suffice.

What is stated therein by the law are only essential determinations of it in the sense discussed in § 4; but to whom, as far as the law is concerned, one may expect to come the nearer the closer the number of values and therefore deviations, on which it is referred, is multiplied. Let us now discuss the same in its application to collective deviations. By convention, § 10, the general expression Θ with respect to *A* can be interchanged with Δ , and ε with η ; but here we stand by the general expressions.

§ 25. The general meaning of GAUSS's law, according to the above hint, is that, assuming a symmetrical probability of the deviations. of the arithmetic mean *A* and a large, strictly speaking infinite, *m* , which is the basis of the derivative of *A* , to determine the relative or absolute number of deviations Θ and hereby deviating *a* , which is contained between given deviation limits, bearing in mind that this determination can be altered empirically by unbalanced contingencies, the smaller the *m* on the basis of the derivative of the *A* and hence the *m*of these deviations is itself. [3)] In short, the GG is a distribution law of deviations and hereby deviating *a* under the above conditions.

[3)] It may also be the case that the *A is* derived from a large *m* , but the distributional relations are studied only for a small number of deviations, but here I abstract from this compound case of little interest to us.

So you have a variety K.-G. in front of which satisfies the requisites mentioned in the previous chapter have from, bemerktermaßen with *a* to be designated, copies the arithmetic mean of $A = \sum a : m$ pulled, the positive and negative deviations ± have Θ of all the individual *a* of *A* taken and from the sum of the Θ without regard to its sign, that is, drawn from its absolute values, the mean $\varepsilon = \sum\Theta : m$, it has, according to earlier explanations, the so-called simple mean deviation. *A,* which applies here as a mean deviation par excellence.

§ 26. In order to explain the application of the law first to its statement for a particular case, we shall find the number of deviations which goes from *A* an, ie from Θ = 0 to a deviationλιμιτ Θ = 0.25 ε or, which is factually the same, which ranges from Θ : ε = 0 to Θ : ε = 0.25, this number is found after a table into which the GG translates, equal to 15.81 p. C. the total number *m* or = 0.1581 *m* , provided that the number is on both sides of *A*followed to the same limit and added together for both sides. For any deviation limit other than Θ : ε =0.25, the same table gives a different relative deviation number; but let us first explain the previous determination by a concrete example.

Suppose we had 10000 recruits, if their *A* and E ηαδ determined the former = 71.7 inches, the latter = 2.0 inches (as is close to the Leipzig student recruitment measures), then assuming that the GG did so 1581 recruits between *A* + 0.25 ε on the one hand and *A* - 0.25 ε on the other hand, which fall between 71.2 and 72.2 inches. In the same sense, let the limit deviation Θ , to which one counts from Θ = 0, be taken as equal to 0.5 ε , hence Θ : ε= 0.5, then, according to the table of the law, the number of deviations from Θ = 0 to two sides at the same time and hence deviating values *a,* ie the number between 70.7 and 72.7 inches, 31.01 p , C. of the total number or 0.3101 *m* . And so, according to the law, there will be a corresponding determination for any value Θ : ε as the limit to which one counts from Θ : ε = 0. Insofar as not all possible values Θ : εWith the corresponding percentage or ratio numbers entered in the table of the law, one finds in a sufficiently executed table those equidistant and so close to one another that one can interpolate between them. The following table, of course, does not give it in a sufficient proximity for exact interpolation, to which one must adhere to a more complete table, but is sufficient for the understanding and the discussions to be drawn here. In doing so, I note that I will briefly call the numbers like 0,1581 and 0,3101 ratios and denote Φ , with Φ [Θ : ε] if, as in the following table, they are functions of Θ : ε are expressed. By multiplying the ratio Φ by the total number *m, in* short by *m* Φ *,* one obtains the absolute number of Θ : ε = 0 up to the given limit Θ : ε . Conversely, if the absolute number between these limits is known, the ratio Φ ισ obtained by dividing the absolute values by *m.*

27. Φ [Θ : ε] table or ε– table of GAUSS's law.

Θ : ε	**Φ [Θ : ε]**	**Θ : ε**	**Φ [Θ : ε]**
0.00	0.0000	2.75	.9718
0.25	1581	3.00	9833
0.50	3101	3.25	9905
0.75	4504	3.50	9948
1.00	5751	3.75	9972
1.25	6814	4.00	9986
1.50	7686	4.25	9993
1.75	8374	4.50	9997
2.00	8895	4.75	9998
2.25	9274	5.00	9999
2.50	9539	5.25	1.0000

In this table, the ratios Φ αρε always given for the output of Θ : ε = 0 up to a given limit Θ : ε . However, in order to obtain ratios for intervals between two different Θ : εin the course of deviations from *A* , say Θ : ε = α and Θ : ε = *β ,* we need only the difference of the corresponding Φ values, that is Φ [β] -Φ [α], which may generally be called φ , according to which z. For example, according to the previous table for the interval between Θ : ε = 0.25 and Θ : ε = 1.00, the ratio to be denoted by φ [1.00 - 0.25] is 0.5751 - 0.1581 = 0.4170 , The following table contains the φ values for equally large, immediately contiguous intervals between the successive Θ : ε οφ the previous εtable from the beginning.

φ– table of GAUSSian law

Successive equal intervals between Θ : ε	φ	Successive equal intervals between Θ : ε	φ
0.00-0.25	.1581	2.75 - 3.00	0.0115
0.25 - 0.50	1520	3.00 - 3.25	0072
0.50 - 0.75	1403	3.25 - 3.50	0043
0.75 - 1.00	1247	3.50 - 3.75	0024
1.00 - 1.25	1063	3.75 - 4.00	0014
1.25 - 1.50	0872	4.00 - 4.25	0007
1.50 - 1.75	0688	4.25 - 4.50	0004
1.75 - 2.00	0521	4.50 - 4.75	0001
2.00 - 2.25	0379	4.75 - 5.00	0001
2.25 - 2.50	0265	5.00 - 5.25	0001
2.50 - 2.75	0179		

These numbers φ are also to be multiplied by the total number *m* in order to obtain the absolute numbers for the respective intervals.

If we denote the Θ : ε οφ the Φ– table, which always starts from Θ : ε = 0 as the first boundary, in short as lim., We see that within small values of lim. the relative numbers Φ the lim. to go almost proportionally; yes you go to a more complete Φ– table, as communicated here, with the lim. to less than 0.25, an even greater approximation to the proportionality takes place, which is within infinitesimal values of lim. can be considered accurate; whereas on ascending to great values lim. the proportionality in question fails completely; and a consequence of this is that in φTable the ratios φ *,* which is the first of the successive equal intervals between the lim. to belong, are almost equal; but the farther one goes, the more rapidly one loses it; as for the equal intervals of Θ : ε from 0 to 0.25; 0.75 to 1.0; 3.0 to 3.25 and so forth are the values *(* φ, 0.1581, 0.1247, 0.0072 and so on).

§ 28. In order to judge the validity and applicability of the GG to empiricism, we must come back to the fact that the assumption of a symmetrical W of the mutual deviations Θ ισ γιϖεν το ιτ . *A is based* on the assumption that, assuming a large, strictly speaking, infinite *m* for each Θ on the positive side, an equally large Θ on the negative side is to be expected; and the ratios Φ and φ are to be regarded as expressions of the W. of the occurrence of the specimens up to given limits of their deviation from *A* or at given intervals of this deviation.

This does not exclude, remarkably, that despite the principle validity of the law under the conditions it presupposes, there are more or less great empirical deviations from its claims, because the condition of an infinite *m* can not be empirically fulfilled. and deviations from its demands can therefore be asserted against it only insofar as the enlargement of the *m* does nothing to bring these deviations closer to disappearance, in short insofar as it does not depend on unbalanced contingencies because of the finiteness of the *m*which are not lacking in clues to be discussed in their place. But let us first follow the implications of the law, on condition that it is of fundamental validity.

In the foregoing it is stated how the ratio Φ and absolute number *m* Φ for both sides together depend on the value ± Θ : ε , to which one follows them to both sides. If this happens only on one side, then, according to the presupposed symmetrical law, the absolute number up to given limits will on each side be half as large as if it were followed for both sides to the same limit of deviation. But inasmuch as the total number of both sides together with large, strictly speaking infinite, *m* reduces to the same symmetrical W. to ½ *m* , the proportions of each side, resp.Φ ′ and Φ , is equal with the total ratio number Φ , whereas the single-sided absolute terms ½ *m* Φ ′ ½ *m* Φ , to assume after the GG for half as large as the reciprocal number *m* Φ to the same limit ± Θ .

Empirically, however, the equality of the two-sided absolute numbers does not apply to the same limit because of unbalanced contingencies; but the GG abstracts from these coincidences and presupposes the case that the difference *m* '- *m* , = *u* vanishes against *m* . It would therefore be wrong, if you ε for the calculation of Φ ′ equal Aθ ': *m* ′ and for those of Φ , equal to AΘ , : *m* , would take, but for Φ ' and Φ , must also as for Φthe value to be calculated from the totality ε = ∑Θ : *m* , since otherwise the assumption of symmetrical W, which is based on the GG, would be contradictory on both sides up to the same deviation λιμιτσ . Also, Quetelet did not put it another way in his comparative tables between calculation according to the Basic Law and observation. Otherwise, of course, where an asymmetrical W. of the deviations. *A* exists, as is actually the case with collective deviations, where the GG is applicable at all only with a further modification to be discussed; but first and foremost, it is important to start from the purely conceived GG itself, and so we pursue its consequences even further.

From the pre statutory symmetrical W . the Θ bez. *A* now follows immediately further that the central value *C,* bez. of which the number of mutual deviations is equal, essentially with the arithmetic mean *A,* rel. of which the sum of the mutual deviations is equal, coincides, that is, that both can deviate from one another only by unbalanced contingencies. For if, according to symmetrical W, on the one hand an equally large Θ ισ to be expected for each positive Θ , the same number of deviations must be expected on both sides with the same sum. But it is the demand that by virtue of symmetrical W, the difference $u = \pm (m' - m_,)$ between the number of positive and negative deviations with increasing m disappears more and more, not to the absolute value of *u,* but to refer to its relation to the total number *m,* di $u: m$,

because u even according to known laws of chance on an enlarged m in ratios of this value but grows against m more so disappears, the larger m , and at

infinite *m*completely disappears. Also, in the absolute growth of u in the ratio of the direction of the difference in itself remains indefinite.

That, assuming the validity of the GG, the densest value D substantially coincides with A , it follows from the view of the φ -table that the number of deviations, and hence deviating values *a,* are greater for both sides for equal intervals, the closer the intervals come to the A , that is, the greatest in the intervals bordering on A , and the same between them, however small.

§ 29. Hereinafter the remark that the table of the GG is not bound to express the limits between which to determine Φ as functions of the simple mean error. In the usual tables, for formal reasons, rather than $\Theta : \varepsilon$, $\Theta : \varepsilon$ or $\Theta : w$ 4) is chosen, which gives tables other than the above, which I briefly referred to as an ε– table, and we will, for the same reasons, be given reasons in the applications to be made in the future rather to a table with reference to $\Theta : \varepsilon$ than the above bez. $\Theta : \varepsilon$ hold; and there you $\Theta : \varepsilon$ usually with t called, I shall such, on t briefly related table t- call table and a running t tell table annexed § 183rd From the very beginning she designed herself for an excerpt from it:

t	$\Phi [t]$
0.00	0.0000
0.25	.2763
0 , 50	.5205
0.75	0.7112

etc

4) [Such a table related to the probable error w can be found at the end of the Berlin astronomer. Yearbook for 1834 (edited by Encke) as Tafel II; in part, it is communicated in § 108.]

Incidentally, such a table is quite correspondingly to be used as the ε - table, as explained in the above example, where A = 71.7, ε = 2.0 inches is assumed. Above all, one has ε with , multiply di 1.77245, are 3.5449 and is now following the *t* - table z. For example, the number of Θ and hence *a* between *A* + 0.25 ï 3.5449 and *A* - 0.25 ï 3.5449, ie between 71.7 + 0.25 ï 3.5449 and 71.7 - 0.25 ï 3.5449, briefly between 72.5862 and 70.8138, = 0.2763 *m* .

The reason for not sticking to the ε– table in the future , which seems to be the simplest, is that an ε– table of corresponding design as the *t*- table does not yet exist, and therefore only for the sake of simplicity the ε– table was taken as the output, which by the way, if carried out, would only have the advantage of omitting the multiplication of ε with everywhere.

A running *t* - table but can be found in different places, eg. B. at the end of the Berlin astronomer. Yearbook for 1834 and quetelet's Lettres sur la théorie des probab. p. 389 flg., In both cases executed only up to *t* = 2.00. A lithographed table available to me, which is no longer in the book trade, gives the execution up to *t* = 3.00 with 7 decimals for Φ [5] . The above ε– table, however, has been obtained from me by interpolation with second differences from the *t*- table as far as it is sufficient and calculated directly for even higher values.

[5] [A corresponding table of equal extent can be found in A. MEYER, Lectures on Probability Theory (German edited by CZUBER), Leipzig 1879, p. 5456549, where *t is* replaced by γ . On the basis of this argument, KÄMPFE has calculated the table published in the Appendix § 183, published in the Philosophical Studies (edited by WUNDT), Volume IX, pp. 1476150, in which the functional values Φ αρεabbreviated to 4 decimals, the arguments *t* resp. Γ however, between the limits 0 and 1.51 are extended to 3 decimal places. A table of appropriate extent with five-digit function values can also be found in the appendix. ó The first table of this kind, to which the said tables are supposed to be the source, has calculated KRAMP, which gives the integrals over exp [- t² *dt* of finite values *t* to *t* = ∞ and the logarithms of these integrals. See: "Analysis of the réfractions astronomiques et terrestres"; par le citoyen KRAMP, Strasbourg, l'an VII, p. 1956206.]

§ 30. Hereinafter I come to the reasons which are the occasion for going beyond the simple GG in the case of collective deviations, as has been explained so far.

From Gauss himself the law is not for collective deviations, as deviations of the individual copy sizes *a*from their arithmetic mean, but noted and noted for observation errors, as deviations of the individual observational values of an object from its arithmetic mean; and in itself nothing less than a matter of course is that a transferability of the law from the latter to the former takes place. In fact, from the very outset, it is very different to have deviations, which are obtained from the arithmetic mean of the measurements because of the lack of sharpness of the measuring instruments or senses and accidental external disturbances in the repeated measurement of a single object Copies of a K.-G. from their arithmetic means for reasons which are situated in the nature of the objects themselves and the external circumstances affecting them.predict *a priori* that nature in these deviations from the means obeys the law of observation errors, but first applied a direct examination of it to K.-G. to do it yourself.

In the meantime, since it was easy to perceive from the outset that in the case of large *m* also in the case of collective deviations. *A* as observation errors the number of deviations *z is*a maximum for a value in a middle part of the distribution board, but from then on decreases more regularly the more the *m* is, and no other law than the GAUSSian, to which one is seeking a distribution law for K.-G. It was natural to think that, above all, it was put to the test. In fact, recruiting measures have been the first item and (with the inclusion of the chest and lung capacity of the recruits) have remained the only one on the other by whom the law has been tried.

This multilateral (by QUETELET, BODIO, GOULD, ELLIOTT and maybe others) [6)] The examination of the measures of the recruits of various countries seemed at first to give everywhere confirmation of the law, in that the deviations from the requirements of the law seemed small enough to be considered insignificant in the sense indicated; In any case, the GG has an approximate validity for recruiting measures, but not so far-reaching as one previously believed to be able to accept, as I have partly convinced myself by critical revision of the investigations thus far conducted, partly by my own investigation of self-procured mulitple recruiting plates there are other K.-G., in which the simple GG fails altogether, while they nevertheless satisfy a generalization of this law.

[6)] [BODIO, La waist of recrues en Italie; Ann. de démographie intern. Paris 1878. GOULD, Investigations on the military and anthropological statistics of American soldiers; United States Sanitory Comission memoirs. New York 1869. ELLIOTT, On the military statistics of the United States of America. Berlin 1863.]

In fact, according to my extended experience, the following two points of view can be given, which make it impossible in the first place, to give the simple GG a general validity for K.-G. concede. The first is this [7)] :

[7)] [The second s. § 34 and 35.]

§ 31. If the GG should be generally applicable to collective deviations, then the implications arising from the symmetrical law of deviations presupposed in the same would have to be deduced. *A* , generally confirm what is not the case, and if recruits and not a few other items remain superficially insecure as to whether unbalanced contingencies or lack of fulfillment of the props are to blame, then other items evade this conjecture decided, as that one essential symmetry of the deviations with respect to Aas a general character of K.-G. could look at. In fact, in its "Lettres sur la théorie des probabilités" p. 166 notes that some K.-G. the difference of the extreme deviations U', U, both sides bez. A constant and legal positive, negative in others than compatible with symmetric probability; and even before I knew of his inquiries about this, I stated with regard to another claim of symmetrical W. that in some K.-G. the deviation numbers bez. A di *mí* and m, not only more constant and legal, but also farther, as can be explained by unbalanced contingencies, differing from one another. Both QUETELET'S and my experience have shown that, depending on the nature of the K.-G. the deviation between U' and U, or the deviation between m' and m, keeps this or that direction; that is, while in size it exceeds the value that might be expected because of unbalanced contingencies, and at the same time in the direction characteristic of one or another type of K.-G. is.

Now I refer to it as an asymmetry in general, when a deviation between U' and U, or m' and m, is composed; but as such will not easily be absent because of unbalanced contingencies, essential asymmetry as such, which can not be made dependent on unbalanced contingencies, is distinguished from insignificant or accidental asymmetry as such, which may be made dependent upon it.

Empirically, the essential asymmetry, even where such exists, mixes more and more with chance, because one always deals with finite m, on which such depends, but since the difference dependent on essential asymmetry in the ratio of m, that of randomly dependent merely in proportion as it grows, the greater the value of m grows, and the more determinate of asymmetry, the greater the value of m, and the greater the value of m, and may itself be regarded as a sign of essential asymmetry the difference found at large $m$between U' and U, or m' and m, the same direction remains with further magnification. In other features but we will later [8] come from, which make it seem no doubt that one in the realm of K.-G. not everywhere with the assumption of mere random asymmetry.

[8] [Comp. in particular Chap. XII "Reasons for Essential Asymmetry".]

§ 32. Now the following alternative appears first.

1) It could be thought that in asymmetry, even where it is essential, only a disturbance of the GG, depending on the nature of the K.-G. to be seen in one or the other sense, which itself does not fit any definite, mathematically formulated laws.

2) It may be thought that the essential validity of the GG for collective deviations from the arithmetic mean remains the rule, but where it is not applicable the cases are to be regarded as exceptions which either come under case 1) or, if indicated, but only exceptionally valid, subject to laws other than GAUSS'schen.

3) Since the deviation between U' and U, as well as between m' and m, at a given $m$insofar as it depends on essential asymmetry, depending on the nature of the

K.-G. different size, and with this the essential asymmetry may assume different degrees, the essential symmetry, where such occurs, may be regarded as the special case of the general case of asymmetry, embracing all possible degrees, where the degree of the latter descends to zero, and could be think that in the area of K.-G. the essential asymmetry represents the general case in its various degrees; the essential symmetry, however, is only a special case, which, if it occurs at all strictly, can only be regarded as an exceptional case, provided the infinitely different possible degrees of asymmetry Disappearance has an infinitesimal W. what does not rule out that the weaker degrees of asymmetry, which may be easily mistaken empirically for a substantial symmetry disturbed only by unbalanced contingencies, are more frequent than the stronger ones, which elude the possibility of such confusion. In relation to this conception, however, it may be thought that there is also a general law valid for the general case, which understands the GG only as the special case, in that the asymmetrical W becomes symmetric.

Which of these three possibilities, and in particular whether one of the first two, which are only modifications of one another, or the third, the more correct one, could not be easily decided, but the decision of the question whether a generalization was necessary secondly, whether the K.-G. suitable for empirical examination, for which the props are specifically indicated in the previous chapter, is really possible in the case of substantial asymmetry according to the same principles by which it is derived for the particular case of essential symmetry; to really submit to the law so deducible. I conducted the investigation on both sides, and both questions were in good congruence in favor of the third alternative. But this includes, of course, an execution of theoretical and empirical investigations, which can not be given all at once and in a short time, but remains reserved for the following chapters, and I only tentatively notice that the most fundamental of theoretical investigations in the nineteenth century. Chapter, the reasons offered by empirical evidence that the presence of essential asymmetry really as the general case in the area of K.-G. be considered, in the XII. Chapters are included. At first, however, it would seem to me of interest to consider the most essential provisions of the generalization of the GG from symmetrical to asymmetric W., hereby from symmetrical to asymmetric distribution at large and only tentatively do I notice that the most fundamental of the theoretical investigations in the XIX. Chapter, the reasons offered by empirical evidence that the presence of essential asymmetry really as the general case in the area of K.-G. be considered, in the XII. Chapters are included. At first, however, it would seem to me of interest to consider the most essential provisions of the generalization of the GG from symmetrical to asymmetric W., hereby from symmetrical to asymmetric distribution at large and only tentatively do I notice that the most fundamental of the theoretical investigations in the XIX. Chapter, the reasons offered by empirical evidence that the presence of essential asymmetry really as the general case in the area of K.-G. be considered, in the XII. Chapters are included. At first, however, it would seem to me of interest to consider the most essential provisions of the generalization of the GG from symmetrical to asymmetric W., hereby from symmetrical to asymmetric distribution at large *m,* to which the combination of theory and empirical research has led me, together present here preliminary beweislos, and although I mention these provisions for several times to be taken out back cover as special laws of asymmetric W. or distribution under special terms as follows on, laws, which one can be satisfied with, as long as a considerable proportionate fluctuation of K.-G. in the sense discussed in (§ 9) gives rise to consideration of another generalization, of which we shall speak later, but which does not lead to a rejection, but only to an intensification of the following laws.

§ 33. Of these special laws, the most important are the first three, which, although set up here in particular, follow from the basic mathematical prerequisites of collective asymmetry in solidarity, as in the XIX. To show chapter. The rest are partly immediately obvious corollaries of them, partly mathematically to deduce from them, as also to be shown later.

Special laws of essentially asymmetric distribution for K.-G. with not too strong relative fluctuation of the same.

1) Basic Law . The deviations are, instead of the arithmetic mean *A,* also to be expected from the densest values *D* which deviate substantially from *A* in the case of significant asymmetry , in order to arrive at a distribution which can be grasped under a simple rule and corresponds to the experience, a rule which, in the case of that the essential asymmetry vanishes, where *D* essentially coincides with *A* , is attributed to the rule of the GG.

2) Two-columned GAUSSian law . The distribution of deviations *In* short, in each case, *D* follows the same rule for each of the two sides, as for symmetrical W. ref. *A is* jointly followed for both sides. It only takes the place of *m,* Θ *,* *ε* = ∑Θ *: m* ρελ . *A* positiverseits *m ',* Θ *', ε* '= AΘ *': m ',* negative hand, *m* , , Θ , , *ε* , = AΘ, *: m* , bez. *D;* With this regard, the same tables, the ε– table and the *t-* table, are still particularly useful for the distribution calculation after each page, as for calculation according to the GG at symmetric W. FIG. *A* would apply to both sides together. Convention now we replace the purposes of § 10 taken to the official designations *m* ' , *m* , , αΘ ', αθ , , *ε* ' , *ε* , which mar. of any principal value, by ***m*** ' , ***m*** ,, ∂ ' , ∂ , , *e'* , *e* , unless it is related to *D* is, so the positive and negative going so proportionate deviation figures Φ 'and Φ , as well as absolute terms Φ ' ***m*** 'and Φ , ***m*** , 'likewise φ 'and φ , ' φ ' ***m*** 'and φ , ***m*** , each on the functions of these designations.

3) Proportion law . The mutual deviation numbers ***m*** ' , ***m*** , bez. the densest value behave like the simple average deviations *e* ' , *e* , , di as ∂ ' : ***m*** 'and ℜ∂ , : ***m*** , bez. *D* , therefore

.

of which are the following corollary.

a) The squares of the mutual deviation figures, di m'^2, $m_,^2$ behave like the mutual deviation sums ∂', $\partial_,$ so:

$$m'^2 : m_,^2 = \partial' : \partial_, .$$

b) The densest value D can itself be determined as the value whose mutual deviation numbers and mean deviations satisfy the law of proportion. Yes, I think this, generally speaking, is not his most convenient but most accurate way of determining, and later (Chapter XI), state how it is to be done. For the sake of brevity, it may be called the proportional, and the Dthus determined , if it is necessary to expressly refer to this mode of determination , be denoted by D_p. This D_p can then be compared with the empirically directly determined D , ie the value to which the maximum of the number $z$falling in a distribution board, comparing it, and finding that it differs from it only within the bounds of insecure uncertainty, find one of the proofs of the validity of our asymmetrical legalism.

4) The distance laws . The distances between the three main values are determined in this way. Is m', the total number, ∂'', the total sum $'' = e\,\partial''\,m''$ the drug of with C or A(whichever one the distance between the C or A studied by D) equilateral deviations rel. D , ie which go to the same side of D , after which C or A Although this may be the positive or negative side, while the index of two dashes below may have the corresponding meaning for the unequal values, according to § 131:

$$C - D = t''e'' \; ,$$

where t''*is* the value of t , which in the table is the t to

.

briefly to Φ''.

a value which according to the proportional law agrees with 2 $\Phi''e''$, as shown in § 131, according to which one can also set:

,

After this, $A - C$ is the difference between the two previous distances:

$$A ñ C = (A ñ D) ñ (C ñ D) = (2\,\Phi'' - t'' \;)\, e'' ,$$

wherein $\Phi''$and t'' are determined as indicated.

5) The π– laws . For the usually occurring case, that the distance of the C of D *has* a small (strictly speaking infinitely small) ratio to the mean deviation e' or $e_,$ the side, after which C of D *is* short, to e'' , one has notably:

Apart from unbalanced contingencies and abnormalities, which in Chap. IV, whereby these relationships, like all laws established here, can be altered, these relations would be strictly valid if $(C - D)^2 : 3 \pi e''^2$ against 1 could be completely neglected, that is, $C - D$ small against e'', but inasmuch as this disappearance never takes place completely, the above πfunctions of D, C, A, or actually have to be substituted:

.

where ξ ισ a positive value exceeding 1 in a small ratio.

The theoretically derivable condition that, assuming a relative smallness of $C - D$ to e'', the value

approximate = ¼ π = 0,78540 belongs to the generality in which it finds itself empirically to the most striking validity of our asymmetrical laws of distribution, and the value p will henceforth be specified in the tables of the elements of the objects treated by me to convince itself of its approximation to ¼ π. An exact correspondence with this is in principle not to be demanded; according to theory, it should result, as noted above, by a trifle greater than ¼ π from the experiments, but this small theoretical preponderance can easily be outdone by unbalanced contingencies, and so it has (for the most accurate proportional determination of$D$ as D_p) in the K.-G., taken from the most varied areas, which could be examined with regard to the validity of the above laws (skull dimensions, size of recesses, botanical, meteorological measures), at the most varied reduction stages and reduction positions of the distribution boards between 0 , 6 and 0.9 found.

Instead of sticking to p, one could also adhere to the two other π - functions, except that because of the smaller ratio, which has $A - C$ versus $C - D$ and completely against $A - D$, these other functions are in stronger ratios of unbalanced contingencies can be affected.

From the third π - equation, which states

.

A very simple way can be deduced to approximate *D* in some other way than directly empirically or proportionally, which is that, having determined *A* and *C* , we find the distance of the sought *D* from *C* 3.66 times takes large, as the distance of the *A* from *C is* found. Soon we may thus determined *D* value as *D* π denote. In the meantime, this provision is too uncertain to even attach any value to it; especially as the laborious determination of *D* as D_P, yet another relatively simple way of very approximate determination as so-called D_i , is available, of which in chapter XI. the speech will be.

In order to obtain, instead of merely approximate, exact determinations of the three distance relations, one has to go back to the exact values of the three distances themselves, which are given under the laws of distance, according to which:

.

.

These relations have two limits, between which they hold, the first of which corresponds to the case ***m*** '' = ***m*** ʺ , that is, to the case of vanishing asymmetry, where ξ = 1; the second case, where ***m*** ʺ , can be set to ***m*** '' vanishingly small, hence = 0. This gives for

1st border: 2nd border:

= *p* 0.7
8540 0.84535

0.21460 0.15465

3,65979 5,46,609.

The value *p* can not normally fall below 0.78540 at all and can not rise above 0.84535.

6) layer law . The central value *C* and the arithmetic mean *A* lie on the same side of the densest value *D* , in such a way that *C* falls between *A* and *D* (see § 134).

7) reverse law . The asymmetry of the deviations *D* has the opposite sign as that of the deviations bez. *A* , di, if *m* '- *m* , rel. *A* (di μ '- μ ,) is positive; so *m* '- *m* , bez. *D* (di **m** '- **m** ,) negative, and vice versa (see § 134). Furthermore, the difference between the extreme deviations has. A , di $U' - U_,$, the opposite sign than the difference between the deviation figures, di $u = \mu' - \mu_,$ (s, § 142).

8) The extreme laws . [Is the number of above resp. lying below D deviations equal m' resp. $m_,$ so there is a likelihood:

that:

$$U' = t'e'$$

represent the extreme value of the upper deviations. Accordingly, the W. is that:

$$U_, = t_, \ e_,$$

the extreme of the lower deviations is equal to:

,

Hereafter, the probable value of the upper resp. lower extreme deviation equal:

respectively. .

If t' and t, *by* means of the t- table:

respectively.

be determined. (See chapter XX)] [9)]

[9)] [The brackets indicate the supplements and additions of the publisher, as already mentioned in the "Introductory Remarks".]

Apart from the π– laws (5) and extreme laws (8), which I first owe to theory, and subsequently found to be empirically proven, the previous laws were first found by me purely empirically, according to which these laws also have an empirical validity ruthless to all theory and can, on the other hand, give confidence to a theory that coincides with it. In vain, of course, a crude determination of primary plates interspersed with great irregularities would make an exact determination of the *D*and to obtain the values associated with it, and seek to gain control over the previous laws; It will therefore still be necessary to discuss how to achieve this purpose by appropriate reduction and interpolation of the distribution tables.

§ 34. It has been expressly stated that the previous laws in the case of not too strong proportionate fluctuation of the K.-G. (in the sense of § 9) can be regarded as sufficient, but demand a further generalization of the GG in the case of a large proportionate fluctuation. Now it has to be stated what can give rise to this, and how to grasp this generalization.

The GSM G. can at infinite in nature itself *m* only be a reaching law and Gauss itself been only explained for [10] ; for it sets no limit to the size of the deviations from *A* on both sides, but only allows the number of deviations to decrease more and more as its size increases. It is obvious, however, that if the deviations from *A* to the negative should be greater than *A*itself, the deviating values *a*less than zero, which is impossible. Thus, the GG can not claim unlimited validity from the outset if it remains valid with the greatest approximation for cases where the deviations from the arithmetic mean, at least the number predominant, remain close to it and on average very small. The same, however, which in this respect applies to the negative deviations from *A* to the pure GG, applies no less to the negative deviations. *D* and the previous generalization and hereby modification of the GG, and there are K.-G., in which the relative fluctuation about *D is* so great that one is no longer sufficient with the previous principle of generalization.

[10] Theoria motus corporum coelestium; Lib. II. Sect. III. artic. 178. Theoria combinationis observ. error. minim. obnoxiae; Pars prior, art. 17; Comment. societ. Götting. rec. Vol. V.

Hereinafter, a generalization of the GG for applicability to K.-G. To distinguish in two directions or in a double sense: 1) unless collective deviations show the symmetric W. attributed to the observation errors with respect to the arithmetic mean, but the case of asymmetry can be considered as the more general, taking symmetry only as a special case comprehend; 2) if collective deviations, even if in the majority of the K.-G., do not show all but the small proportionate fluctuation around the principal values, which belongs to the observation errors.

Now that the K.-G., in which one generalizes the GG in the first direction, is not only far more numerous, but also much easier to treat than those in which it is necessary, the still further generalization in the second In anticipation of the generalization, in the first place, the presentation of the principle of generalization is facilitated in the second sense, but this anticipation has taken place here, and yet, in order to give our inquiry the necessary generality, From the outset, there are two points of view that give the idea a direction as to how this generalization can be grasped.

§ 35. So far we have always kept in mind only arithmetical deviations with respect to any principal values, that is, which may be conceived as positive and negative differences, and usually such as will be further here understood by deviations par excellence. I call them stated to generally Θ . But one can also speak of deviations in relation to given principal values, ie ratios in which a given principal value *H* *will* exceed or surmount *H* , which we generally wish to denote by ψ . So if $\Theta = a - H$ *is* an arithmetic deviation, then $\psi = a: H$a ratio deviation , and while we distinguish Θ 'and Θ , as positive and negative arithmetic deviations, depending on $a > H$ or $< H$, we distinguish from the same viewpoint ψ' and $\psi_,$ as upper and lower ratio deviations .

While strong arithmetic deviations from a principal value down to negative below the size of the principal value, and thus become impossible, this does not apply to strong lower ratio deviations, which, however, as far as they go downwards, only lead to smaller fractions of the principal value which, however, remain just as positive as the principal value to which they refer; because negative ratio deviations do not exist at all, but only positive, which exceed 1; and those which do not reach 1 (as true fractions). After which it could be remembered that the distribution law, in order to deal with comparatively strongly fluctuating K.-G. down to be as applicable as on weakly fluctuating,

But with this mathematical point of view the following empirical coincides in the same direction.

Observational errors are, generally speaking, at least with respect to the measurement of spatial lengths, substantially independent of the size of the object to be measured, insofar as their size does not change, assemble, or complicate; for, of course, the errors of observation when measuring a mile will be greater than when measuring a foot length, but only because more and more complex operations belong to the measurement of the former; However, the observation error in measuring a high thermometer or barometer generally speaking are not greater than when measuring a low.

Against this, K.-G. generally in substantial dependence on their size, if understood in the sense of the following examples. On average, a flea is a small creature, and so the deviations of each flea from the middle flea are on average only small, only fractions of its mean size, and the whole difference between the largest and smallest flea remains small. The mouse is on average much larger than the flea, the horse again much larger than the mouse, a tree much larger than a herb, etc., and everywhere a corresponding remark returns. The deviations of the individual mouse specimens from the middle mouse are on average greater than those of the individual flea specimens from the middle flea, etc. Also, this dependence of the average size of the variations on the average size of the object can be understood from the fact that the internal and external changing causes find more targets on large objects than on small ones. To be sure, the quality of the objects, too, is influenced by the greater or lesser ease with which they give way to the changing influences; furthermore, the accessibility to external changing influences may vary according to circumstances. Thus, a precise proportionality of the average size of the deviations with the average size of the objects is not to be expected from the outset. But anyway, the size of the objects remains a major factor in the size of their changes, and even if their average size at different K.-G. Deviations refer.

§ 36. First of all, the apparent difficulty arises from this idea that the GG is, by its very nature, only derivable from deviations that can be grasped as positive and negative differences from their initial values, and hence can not come under a law as a special case to ratio deviations, and yet we seek a law which, in the case of vanishing asymmetry and weak relative fluctuation, passes into the GG or reflects its distribution. But let us translate the ratio deviations $\psi = a : H$ into their logarithms, $\log \psi = \log a - \log H$, which we briefly describe as logarithmic deviations with λ may indicate, and note:

1) that the logarithmic deviations $\lambda = \log a - \log H$ divide the character of the arithmetic Θ, to allow themselves to be taken as positive and negative differences from a given initial value, except that this itself is a logarithmic, no longer H, but $\log H$;

2) that, as long as the arithmetic deviations are comparatively small compared to their principal value, that is, a relatively small fluctuation around them takes place, as is assumed in the GG, the ratios of the arithmetic deviations significantly coincide with those of the associated logarithmic, which is not only mathematically provable, but also empirically on the logarithmic tables can be demonstrated by comparing the differences of the logarithms with those of the corresponding numbers.

Thus, even in the case of comparatively slight fluctuation, we may use advantage of the logarithmic principle as the most generally acceptable one, except that this advantage, in comparatively weak variation, is too small to be worth the increased effort which the logarithmic treatment brings in the case of comparatively great fluctuation, it will clearly show what the empirical evidence will follow; for, of course, without empirical evidence, the previous conception could only appear as a hypothesis built into the air. The application of logarithmic treatment to empiricism, however, is this.

Reduce the given individual measures *a of* the K.-G. look for their logarithms $a = \log a$, in the same way as when the densest value D comes from the a, we look for something more specific, the densest value of this $\boldsymbol{a}$, which is called $\boldsymbol{D}$, and which, as explained later, is not to be confused with $\log D$, take from this value $\boldsymbol{D}$ the logarithmic deviations $\lambda = \boldsymbol{a}$ ñ $\boldsymbol{D} = \log a - \boldsymbol{D}$Which partly positive and partly will be negative, seeking from λ to each side, in particular, di λ' and $\lambda,$ the simple arithmetic mean or so-called mid-log deviations. E', $e,$ respectively:

, ,

wherein $\boldsymbol{m}'$ and $\boldsymbol{m},$ the number of positive and negative deviations, not as used to be the *a* of *D,* but of $\boldsymbol{a}$ of $\boldsymbol{D}$ mean, and then determine the distribution of the logarithmic deviations λ', $\lambda,$ on each side, in particular also in relation to e', $e,$, $\boldsymbol{m}'$, $\boldsymbol{m},$ after zwiespältigem GG, as indicated under 2) above (§ 33), except that e', $e,$, $\boldsymbol{m}'$, $\boldsymbol{m},$ here logarithmic in the given way, instead of arithmetic determined in the past.

Provisions for the deviations in relation and their principal values then follow from these determinations which apply to the logarithmic deviations by translating them into the numbers belonging to the logarithmic tables, but which we shall not consider for now, since the necessary explanations are reserved for a later chapter, which is generally reserved for the logarithmic treatment of K.-G. more detail (chapter XXI).

Apart from the logarithmically denominated value $\mathbf{D}$, we can then obtain the logarithmic mean $\mathbf{G}$ as $\sum \mathbf{a} : m$, that is, as the arithmetic mean of the logarithms of *a,* and the logarithmic center $\mathbf{C}$, as the value of $\mathbf{a}$ **,** which has the same number $\mathbf{a}$ above and below it , determine.

From the logarithmic values, one can also pass to the numerical values which belong to them according to the logarithmic tables, and set special terms for them, which is not idle, since these values have their notable importance. Thus, the numerical value belonging to $\mathbf{D}$ can be denoted by $\mathbf{J}$ as the closest ratio value , since it has the meaning that at equal distances from it there are united more values *a* and consequently $\mathbf{a}$ than at the same ratio distance of any other *a.*

The numerical value associated with the log center $\mathbf{C}$ coincides with the arithmetic C ; for if a value of a , di C , is equal to a above and below itself, then the logarithm of C , di $\mathbf{C}$, has the same number of logarithms of *a,* ie equal to $\mathbf{a}$, above itself and below itself.

The G , which belongs to $\mathbf{G}$ as a numerical value , represents the geometric mean of a .

§ 37. Thus, we have to distinguish the following three general laws or principles, each of which may be regarded as a generalization and at the same time an intensification of the preceding, and whose essential differences are to be briefly summed up here.

1) The pure, simple, original GAUSSian law or principle, for the presupposition of symmetric probability of the mutual arithmetic deviations Θ', $\Theta_,$ from the arithmetic mean. Here, the output is taken from the arithmetic mean A , the mutual deviations determined as arithmetic, the mean deviation $\varepsilon = \sum \Theta : m$ for both sides together as the quotient of the sum of the mutual deviations by absolute values by the total number of the same directly (or after a known formula from the sum of the deviation squares as) and after the t Table determines the distribution. To explicitly distinguish the relation of the deviations on A , I substitute the general terms m , Θ , ε by μ , Δ , η .

2) The arithmetic generalization of the GG, for the presupposition of asymmetric W of the deviations Θ', $\Theta_,$ from the arithmetic mean, generally valid for the most different degrees of asymmetry, but only sufficient for relatively weak variation around the principal values, as most K .-G. due. Here the output from the arithmetic densely values D taken from the dimensional values a in later contemplative manner [11)] is obtained without having them previously translated into logarithms. The mutual deviations Θ', $\Theta_,$ are as arithmetic to both sides of$D$ particularly taken, their mean values $\varepsilon' = A\Theta' : m'$ and $\varepsilon_, = A\theta_, : m_,$ determined, and then for each page in particular, the distribution according to the two-column GG (§ 33) after

setting of $t' = \Theta' : \varepsilon'$ for positive side and $t_, = \Theta_, : \varepsilon_,$ negative side of the t table determined. To explicitly distinguish the relationship of the deviations on DI replace the general terms m , Θ , ε with $\boldsymbol{m}$, ∂ , e.

[11] [p. Cape. XL]

3) The logarithmic generalization of the previous law or principle, valid for arbitrarily large asymmetry and arbitrarily large proportionate fluctuation. Hereinafter are all single dimensional values a logarithms $\boldsymbol{a} = \log A$ to take therefrom the densest value $\boldsymbol{D}$ to determine logarithmic variations λ', $\lambda_,$ to take on both sides thereof, the means of the same e', $e_,$ to take and on $\boldsymbol{a}$, $\boldsymbol{D}$, λ' , $\lambda_,$, e' , $e_,$ apply all relevant provisions as according to the previous one, the arithmetic generalization to a , D , $\partial' \partial_,$, e' , $e_,$. The logarithmic values can then be used to arrive at the ratio values as numbers belonging to the logarithmic tables.

In principle, strictly speaking, I merely look at the logarithmic generalization of the GG, ie 3; but in its application it is very cumbersome, and in the case of comparatively weak variation one can very well proceed according to the arithmetic generalization, as experience has shown. The least commonplace is the simple GG 1), while it is easiest to apply because the arithmetic mean *A is* easier to determine as the initial value of the deviations more easily than the densest D and $\boldsymbol{D}$ values with relative accuracy; but with weak asymmetry the results of 1), 2) and 3) differ little from each other.

Depending on whether I now treat the treatment of an object on the assumption of symmetrical deviation. *A,* according to the first principle, or assuming asymmetric W. bez. *A* , that is, according to the second or third principle, I will briefly speak of treatment according to a symmetrical or asymmetrical principle; and I will speak of arithmetical or logarithmic treatment, depending on the treatment with the use of arithmetic deviations, that is, according to the first or second principle, or with the application of logarithmic deviations, that is, the third principle.

In general one finds for the following the treatment of the objects and arrangement of the sentences according to the arithmetic principle; However, the transition to the logarithmic principle and the treatment of such essential objects is especially reserved to Chapter XXI.

VI. Characteristics of the collective objects through their determinants or so-called elements.

§ 38. Let us go to the earlier ones (chapter II) concerning the characteristics of the K.-G. now made something general.

Should a K.-G. If it is completely determined by measure and number, then it would even be the case to count not only all present but also past and future copies of it, and to take from each the measure according to the considerations which give space to a quantitative determination, as if by size the three main dimensions, weight, tightness, duration. This is generally impossible. The quantity of specimens of a given object is generally indefinably great, and of this indefinitely large quantity there is usually only a very limited number of measures to be taken. It is clear that if z. For example, comparing the brain weight of the European and Negro, this can not be done by contrasting the weights of a thousand European brains with the weights of a thousand Negro brains. There is a uniform result. Thus, it will be true, according to earlier remarks, to measure as many specimens of the objects to be examined and compared as possible without the arbitrary exclusion of certain quantities in which one can not do too much so as not to give too much room to unbalanced contingencies in the manner indicated, they are to be classified according to number and size in distribution boards, and since this, however, first causes the generality of the values to be overlooked, from these distribution boards certain values, the so-called determinants or elements of the K.- G. derive a characteristic of the object and possibility of its comparison with other objects by quantitative relationship.

If one suffices, as is often the case, with the indication of the arithmetic mean of a K.-G., then one has therein an important and in no case negligible determination value and comparison value with other objects; but there can be two K.-G. to agree completely or closely in it, yet diverging very much according to other relationships. Now it could appear soon enough, including the mean fluctuation size and the entire fluctuation range of a K.-G. by taking into account the mean deviation from the arithmetic mean and the extremes, in order to have exhausted the essential characteristic, and indeed this has sometimes happened. But with the knowledge of the K.-G.

§ 39. If now the general collective gauge can not stand still in the formerly familiar, limited consideration of *A* and the related deviations, and yet, as already admitted above, not every K.-G. It will not be easy to consider the consideration of all the possible determinants given in Chapter II, unless they are taken into account in a collective law which is a very special one Attach importance, and should serve as an example of the feasibility of universal consideration itself. So you may want to have a guiding view for a selection to make.

All in all, now I believe that where one wants to save with determinations, and there is a convention of which principal value one prefers to the characteristic distinction of given K.-G. but that the arithmetical mean with its deviations will always remain the advantage it has been granted to date, only that with overriding of the other determinants one at once loses insight into the quantitative constitution of K.-G., and disregards characters of the same, which in themselves no less important than those who make use of the arithmetic mean, and to lift up the establishment of a general law of distribution. To clarify this point, we must return to the characteristics of the various principal values given above (Chapter II) with an expansive and illustrative consideration.

[This is explained in detail in X. Chap. happen. But while there the properties of each main value are presented for themselves, this is a comparative assessment of the main values themselves in terms of their performance on the characteristics of the K.-G. For this reason, only the arithmetic mean *A,* the central value *C* and the densest value *D come* into consideration; because the separation value *R,* as well as the heaviest value *T* and the deviation value *F*are from the outset because of their lesser importance in a selection to be made to leave aside. However, there is a difference between these three main values, with regard to a distribution law assumed to be valid or disregarded, since a very different appreciation of the same applies to the same place.]

§ 40. [If one lets fall the premise that a distribution law regulates the course of the *z* -values of a distribution chart, then the latter is to be understood in principle only as a random accumulation of values, and it can therefore only the meaning to the main values, as Means to summarize and represent that random complex in a more or less appropriate way. But then there is no doubt that the determination of *A is more* valuable than that of *C* or *D.* For *A*represents as an arithmetic mean the average value that can actually be substituted for each individual value if they are to be combined into a sum. *C,* on the other hand, merely indicates the value center, which is exceeded as often as it is undercut, and thus represents the values of the table with less reliability, because it does not depend, like *A,* on the sum, but only on the number of mutual deviations. *D* finally, it can not be admitted as a representative mean value, since it denotes only the empirically most dense value in its randomness governed by no law, and can not be determined mathematically in its position, but can only be found by the sight of the table. In fact, its actual presence in a randomly drawn panel can only be regarded as a happy coincidence, to which no importance attaches.]

[It is different if the existence of a distribution law is assumed. Then *A* retains the mean value that it has in the random table without directly gaining anything. On the other hand, the meaning of *C* becomes larger, since it represents the probable value as the value center, taking into account the probability concepts which now come into force. *D,* however, comes into the center of attention because, as the empirically closest value, at least approximately, that is, apart from the unbalanced contingencies, it denotes the value to which the greatest W. belongs. *D*thus stands in solidarity with the distributional law, whose maximum value must principally coincide with it. It is also evident that, after the establishment of an appropriate law of distribution, a double way of determining *D is* open: the one on the basis of the law, whose maximum value theoretically denotes the most probable value; the other on the basis of the table whose closest value empirically indicates the most probable value. It does not matter whether the passage of the *z* in the table indicates the densest value directly or only the tendency to produce it. Because as a result of the law that came into force, the *a* and the *z* in functional connection, so that, according to known rules, the closest *z* can be calculated by interpolation from the given tabular values, if its crude determination from the direct sight of the table fails or appears inaccurate. Insofar as this empirical determination of the most probable value agrees with the theoretical one, the *D* must be given all the properties which characterize the maximum value of the law of distribution, so that partly the calculation of the *D* by interpolation offers a means of attaining the validity of an established law of distribution On the other hand, before being aware of the law to be established, the knowledge of the properties of the empirically ascertained *D is confirmed* the tablets can give hints to find a law of distribution.]

§ 41. [This solidary connection between the properties of the dense value *D* and the distributional law, which assures the *D* the unconditional precedence over any other principal value, also appears in the physical and astronomical error theory. It is known to regard as the true observation value the arithmetic mean of the observed values whose deviations from those are the observation errors. The but true value is nothing other than the most probable value, which in a series of errors sufficiently large to reveal a lawful course, is to be recognized as the empirically most dense value. Thus, by establishing the principle that the true or most probable value is the arithmetic mean *A* , *A is given* the meaning of being the densest value *D* at the same time . This requirement of the fundamental coincidence of *A* and *D* now leads to the GAUSS error law, as the z. From ENCKE's [1)]Representation of the method of least squares can be seen. On the basis of this, the fundamental agreement of the central value *C* with *A* and *D* , whose combined position for the course of the panel symmetry, follows. *A* conditionally, while their divergence results in asymmetry.]

[1)] [Berlin Astronomical Yearbook for 1834, p. 264 fg.]

[That principle, of course, must be confirmed by experience. This does not require, however, that for error series whose extent enables us to give a most dense value by the direct sight of the series or by interpolation calculation, it coincides exactly with *A* ; for one will always have to take account of unbalanced contingencies, which may cause an empirical divergence of the principal values without, at the same time, questioning the validity of the established principle. Moreover, a probation of the principle will rather be found in the agreement of the course of values actually present in the series of errors with the course required by the law, than in the empirical coincidence of *A* and*D* seek and find; as well as z. B. BESSEL in the "Fundamenta astronomiae" by confronting the course of the errors according to the theory and after experience has given a probation of the GG. The unbalanced contingencies, especially with a sufficient reduction in the error table, will in general have little effect on the course of the table values, while it is expected that they may sometimes disturb the position of individual values, and a comparatively great divergence of the principal values, their coincidence, from the Theory is required to cause.]

But insofar as such a divergence takes place, the arithmetical mean retains the advantage that, according to Gauss's principles, the most probable value is considered to be that of which the sum of the squares of deviation is the smallest possible, or in relation to this the sum of the deviations after both Pages is the same; but both values coincide in arithmetic mean, whether symmetry or asymmetry take place with respect to it. Thus, the preference for the arithmetic means remains, even where it does not coincide with the other principal values, in the physical and astronomical standard, according to their purposes.

[This is true only on the premise that, in principle, the arithmetic mean should be regarded as the most probable value. If this principle loses its validity, then *A* also loses his privileged position; for it retains its original meaning as an average value, but in consideration of the law of distribution, it is replaced by that value which assumes the role of the most probable value according to the principle to be established, and in principle coincides with the densest value. For example, if the central value *C* or another "power mean", with respect to their establishment and discussion on the treatise [2)]"To refer to the initial value of the least deviation" is regarded as the value to which the greatest W. is supposed to arrive; in connection with it, a different law of distribution comes into force each time; by its existence the most probable value on which it is based is quite similar Supremacy receives the arithmetic mean as in the validity of the GG.]

[2)] Treatises of the math.-phys. Class of the royal Sächs. Gesellsch. the science Volume XI **,** 1878. (In particular, Section VI: "Remarks on the validity of the principle of the arithmetic mean" and Section VII: "Probability laws of deviations with respect to the various power resources, subject to the validity of their principle.")

§ 42. [For collective measurement, the densest value is of fundamental interest in the same way, as soon as the distribution of the copies of a K.-G. dominant probability law comes into question. With regard to the determination of the properties of the denser value and the derivation of that law, which can be grounded on them, the principle of the arithmetic mean, or any other principle *,* can not be *a priori*be set up. Because the K.-G. are only given by experience, and there is not even certainty from the outset that they will find a definite value as the most probable value, or that, in other words, the empirically denser value in the various K.-G. can be characterized by the same properties. It is, therefore, to be regarded as a fundamental result of the experience that the most diverse K.-G. which have been examined in fact permit the determination of a most probable value, and that the latter coincides closely enough with that for which the Ratio of the mutual mean deviations (*e ': e* **,**) is equal to the ratio of the mutual deviation numbers (***m ' : m ,***). The densest value is thus Kollektivmasslehre of the arithmetic mean in principle different and is rather in accordance with the principle of required by the proportion*e ': e***,** = ***m** ': **m ,*** defined values. The latter (which, according to the definition given in Chapter II, isD_p, whileD_i naming the interpolation-calculated, empirically denominated table value) claims the same respect here as the arithmetic mean in error theory. He also has the corresponding meaning; because on the principle that the most probable value of a K.-G. the proportion *e* ': *e* **,** = ***m** ': **m ,*** or that $D_p = D_i$ is to be found as distribution law, the extended GG already predisposed in the previous chapter in a similar way, as on the principle that the most probable value the arithmetic mean, or that $A = D_i$ is to be the simple GG results as an error law.]

[Only in this respect can A assert supremacy here, too, as is the case with K.-G., who is gifted with weak asymmetry. so closely coincident with D_p that it suffices to approximate the simple GG instead of the two-column one.]

§ 43. When choosing between the different principal values, the degree of lightness and determination with which they can be won must not be ignored. If it is a question of crude determination, that of the most dense value is decidedly the simplest and easiest, since in a distribution chart one needs only to look for the a to which the largest z belongs; soon in this regard, the determination of the central value, for which there is only an enumeration which follows a or Θ from both sides towards the center to the obtained equality of m' and $m_,$, needs; most cumbersome of the A, since the addition of all individual a of a manifold distribution table, or what comes to the same thing, the formation and addition of the products za to obtain the sum $\sum a$, which is to be divided by m, is a tedious and tedious operation at large m.

But otherwise, and vice versa, the relation arises, if one wants to go to sharp, the ideal approaches approaching as possible. From the raw determination of the densest value to the maximum z falling on it, only a very uncertain approximation to the ideal value is to be expected; but the strongest possible, to the relationship $m' : m_, = e' : e_,$ to be founded, although to bring to a certain and not difficult to bring into account, but in the execution becomes unstressed, reduction and interpolation, the last still leave a small margin for the result to be calculated. Also the sharp determination of the C although much simpler than that of the D, can not do without such aids, whereas the determination of the A does not need such. The complexity of the formation of the products za can be avoided by a later (chapter IX) to be specified method.

§ 44. After the preceding discussion of the qualities and achievements of the various principal values, there will still be something to be said of the points of view from which the extremes and deviation functions come into consideration.

There can be two K.-G. The fluctuation range and the mean fluctuation value of the specimens may be very different from one another by their principal values, which are by no means indifferent distinguishing features. Thus, the mean temperature of an island in the middle of the ocean and a location in the middle of a continent can be the same; but the deviations of the individual temperatures from the mean temperature are narrower at the first one and are on average smaller than at the second, after which we distinguish sea climate and continental climate.

[It will now be inclined such differences by specifying the largest and the smallest value, that is, the E' and $E_,$ which in a number of copies of a K.-G. to characterize in the simplest way.]

However, while it is advisable to state the extreme values E' and $E_,$ in order to reveal the limits within which the size of the specimens has fluctuated, the usefulness of more than one relationship is precarious and limited. Once these values are subject to great contingencies, so that one can not count on finding the extremes and extreme variation of a new series of specimens with the same *m* to find the same values; secondly, the statement of these values has a value only for the number of specimens, *m* , from which they are derived, since the latitude of *m increases* the latitude of the changes, so that for larger *m*generally further apart extremes, a smaller *E* , 'a larger *E',* and thus a greater extreme variation *E* '- *E* , than at smaller *m.* Now set z. For example, one wants a measure of the absolute and relative variability of a K.-G. in which values *E* '- *E* , or (*E'* - *E* ,) *: A* Search, as is done well, and after several K.- G. compare, so you will commit the greatest errors when the objects a different *m* have , and I have really met with errors of this kind, which also led to erroneous conclusions.[3)]

3) [This paragraph is taken from an exposé by FECHNER on average deviations and extremes, which in 1868 was communicated to Prof. WELCKER and made available to me by him.]

Better than the fluctuation range *E* '- *E* , therefore, the mean variation, identical to mean deviation , is suitable for the measure of the variability of an object, since it is quite independent of *m*is and can be made completely independent of it by a suitable correction. However, this measure changes according to the principal value from which the deviations are calculated, and, generally speaking, is different for the positive and negative sides. The consideration of the latter difference, however, escapes if the total sum of the deviations on both sides, divided by the total number of deviations on both sides, is used everywhere, that is, according to our general designation as a mean fluctuation or average deviation par excellence with respect to a given principal value:

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Whether you want to use the deviations of one or the other main value depends on which one you want to refer to, and one thing does not exclude the other. As you can see, for a given *m* the measure changes according to the total sum of the mutual deviations with respect to the different principal values; until now we have merely made use of the deviations of the arithmetic mean, and if we stand by it, we obtain as a mean fluctuation value in the sense of the above designation:

,

However, η ισ not completely independent of the size of *m* , but it behaves like this: The value *A* , from which the deviations are taken, changes somewhat according to the number of *a,* and hence the *m of the* same, from which he forms the mean ; and the most accurate *A* could only be obtained from an infinite *m* . With the size of the finite *m,* that is to say, in any case inaccurate *A* , the size of the deviations, and thus the sum of the same, by whose division *m is* obtained with the value η , teaches theory and experience[4] that ∑ Δ and thus η = AΔ: *m* with increasing *m* average of the ratios is growing, after which ∑ Δ , and η in the normal case, that the determination of *A* with its deviations from an infinite *m* would have happened , may return, by ∑Δ resp. η with , noticeably = 2 *m* : (2 *m* - 1), multiplied, which is called the correction because of the finite *m* . That correctedη ισ called η_c , and thus finds itself:

,

[4] In both respects, compare my essay in the reports of the King. Saxon Society of Sciences, Volume XIII, 1861 ["On Corrections Regarding Accuracy Determination of Observations, Determination of the Variation of Meteorological Individual Values about their Mean and Psycho-Physical Measures on a Mean Error Method"].

Although this correction does not apply in every single case, but in the average of the cases, and since one has no means of accurately determining them for each individual case, one must adhere to the value that applies in the average of the cases, and therefore, if one does not shy away from the small amount of correction, it is better to adhere to η_c than to η in the collective theory of measure .

If the mean variation with respect to *C* or *D is to be* determined, then without correction one obtains ε = ∑Θ *: m at first* , and *e* = ∑∂ *: m* in the σεχονδ χασε , but the correction would remain the same, as far as I overlook it. The mean variation with respect to *C* has the interest that it is smaller than with respect to *A* and *D* , the smallest possible, because according to the earlier statement, the sum of the deviations with respect to *C* is the smallest possible, and this translates to its quotient by *m* .

Generally speaking, although this may be exceptions, and exact proportionality does not take place, the mean variation increases with the size of the objects, and so it may be of interest to eliminate this influence as much as possible by taking the mean variation divided by the size of the fluctuating object, hereby takes into account the relative but the absolute fluctuation.

§ 45. More important than the measure of the fluctuation of an object about its principal values, the mean deviation derives as a middle term for the determination of the distribution of the object. The physical and astronomical doctrine makes use for this purpose of the mean deviation ε with respect to *A* or the values relating to ε

, but this is permissible only for the symmetrical errors of observation which are presupposed in this doctrine, whereas the collective theory of measure is valid for them actually more general assumption of asymmetry only from the mean deviation with respect to *D*, and not jointly for both sides, but each page can make particular use (see § 33), ie of:

,

Here, too, strictly speaking, a correction is to be made because of the finite *m* ; but the corrected values are not what you might think to put:

, ,

rather:

.

In fact, otherwise, the correction of the two sides with respect to the sums of deviation would not be in agreement with the common correction of the total sum of them.

For the total sum you have namely:

,

If you wanted to set especially for the mutual deviation sums:

, ,

such would be obtained by summing these values:

.

which is not correct with the above values for $\sum\partial_c$.

§ 46. Finally, there are still some values to be commemorated, which are related to the very important asymmetry rules which have been repeatedly discussed, but which will be discussed in detail later. For the time being only the following about these values.

It is first the difference $\mu' - \mu_, = u$ between the number of positive and negative deviations of *A* and the difference $U' - U_, = (E' - A) - (A - E_,) = E' + E_, - 2A$ between the magnitude of the positive and negative extreme deviation of *A,* which come into consideration in this regard. More importantly, but as these absolute differences are the relative:

and .

Here, for the time being only, in consideration of the use to be made of it later on.

Of a difference between the sum of positive and negative deviations of *A,* di AΔ 'and AΔ , of course there can not be, since *A* is specifically designed so that both sums are equal; but this does not yet imply that at the same time both deviations μ', μ, become equal to each other, and at most by chance one will find it. But what you regarding general or only with random exception, at least on average, the collective deviations *A* place is that $\mu' - \mu$, with the size of *m* increases.

Assuming equal W. positive and negative deviations namely the probability theory teaches to recycle of the case to the box with the same number of black and white beads that $\mu' - \mu$, its absolute value according to an average of the ratios of increase. However, the more *m* increases, the smaller the ratio of :

m becomes , so that at infinite *m* , zero and one become.

One consequence of this is that in the later investigation, the positive and negative deviations *A* have an equal W. really not the absolute difference simply *u* may hold that is not generally lack even with the same W., but on his relationship with *m* , which must not exceed a certain size should gleicheW . not very unlikely, about which will be more later.

So far we have the inequality of the mutual number of deviations bez. *A* di μ', μ, taken as a feature and in some respects as a measure of asymmetry. Of course, from an asymmetry due to inequality of the variance ΣΔ ', ΣΔ , bez. *A* no question, because it is in terms of *A* is that AΔ '= AΔ , ie, *A* has to be determined so that this equality occurs; On the other hand, a feature or measure of asymmetry could not indicate an inequality in the number of deviations. *C* because it is in the concept of *C* that the mutual number of deviations in relation to it is the same; against that nothing would prevent, rather than the asymmetry with respect to the arithmetic mean value per se *A* on the densest value *D* after the inequality of the deviation numbers **m** ', **m** , to determine, in the case of both main values differ enough from each other; with the advantage that, with respect to *D,* a greater divergence of the deviations **m** ', **m** from one another than the deviations μ', μ *is* due to the laws of asymmetry, bez. *A get* from each other; and the **m** ', **m** , can be related to the bilateral G. G, while in the case of asymmetry against *A,* neither the single nor two-sided GG is valid anymore with respect to the deviation number of *A.* It should be noted that if bez. *A* μ' over μ, overlaps, conversely **m** , over **m** 'overlaps. However, since *A* and hereafter μ', μ, are much easier to determine than *D*and hereafter **m** ', **m** , and of a greater or lesser asymmetry. *A* always on a larger or smaller, only in each case the asymmetry bez. *A* exceeding inequality bez. *D* can be deduced from the opposite direction, it seems generally more practical to first look at the results of the determination of the asymmetry by $\mu' - \mu$, ref. *A* , in so far as it implies the inequality of m' and m, rel. *D*can be closed; However, if it is a question of exact determination, it must be examined in particular according to theory and empiricism.

VII. Primary Distribution Charts.

§ 47. [In the preceding chapters, the main points of the investigation were pre-emptively explained. Now it is time to actually conduct the investigation. Since it is not based on hypothetical assumptions; but based entirely on experience, it can only be empirically given by K.-G. go out yourself. The latter, however, in their original form are neither suitable for derivation nor for the proof of the theoretically valid laws. Above all, their mathematical treatment must be taught. On the one hand, it deals with the preparation of a form of analysis suitable for examination by setting up primary and reduced distribution boards (Chapters VII and VIII); on the other hand, it gives rules for calculating the principal values and deviation functions, in which the characteristic features and properties of K.-G. to present oneself (Chap. IX - XI). For the sake of simplicity, only the arithmetic treatment of K.-G. be the talk; for the logarithmic treatment, with which only the full generality of the method of investigation is attained, agrees with the arithmetic of the main thing, in that only the logarithms of the measures take the place of the measures themselves.]

[Now that a suitable document has been obtained for the theoretical investigation, the task is first of all to present the asymmetry of the K.-G. and to establish criteria for distinguishing essential and non-essential asymmetry (Chapters XII - XVI). In that case, however, the laws of partition valid for essential symmetry and essential asymmetry must be developed (Chapters XVII - XX). In this case, the usual case of a small proportionate fluctuation of the individual values around the principal values is assumed.]

[This main part of the investigation is followed by a discussion of the modifications that are caused by the transition to the logarithmic distribution law. Logarithmic treatment requires primarily the K.-G. with strong relative fluctuation, but also the relationships between the various dimensions of K.-G. require such (chapter XXI and XXII). Finally, the dependency ratios of the K.-G. discussed (Chapter XXIII).]

§ 48. [If one wants a K.-G. In the first instance, the individual specimens of the same are to be measured in the random, spatial, or temporal order in which they present themselves, and the measures to be designated by *a are* to be listed in a primary list. Care must be taken to ensure that the props specified in Chapter IV are met, ie in particular that a sufficient number of measures are brought together, excluding any abnormalities.]

[As already mentioned (§ 3), such an initial list is not yet suitable for arithmetical treatment. However, it is valuable in other respects, since it allows the determination of whether the copies of the K.-G. vary independently of each other or are in a dependency relationship. In this regard, rules were specified in § 20, which are described in chap. XXIII a further embodiment can be obtained. In the interest of mathematical treatment, however, one must order the dimensions of their size and hereby produce a distribution panel from the original list. It is referred to as the primary distribution panel to distinguish it from the reduced panel, whose preparation and treatment is taught in the next chapter. In the same form the dimensions *a*a column progressing from the smaller to the larger values, containing each *a* only once, while an enclosed column lists the corresponding numbers *z* , which indicate how often each *a* occurs.]

[This primary panel is now the starting point of the whole investigation. However, it is usually still subject to strong irregularities and usually has such an extent that its communication would take up too much room. One will therefore seek to counter both disadvantages by making reductions, and then generally restricting themselves to the demonstration of the panel in its reduced form. But this is about getting to know the nature of the primary panels and gaining an insight into the peculiarities that may arise; therefore four of them, serving as examples K.-G. the primary panels are presented.]

§ 49. [The first two panels I and II give the measurements for the vertical and horizontal circumference of 450 European male skulls. It should be noted that the here and in the following consistently held designation "vertical circumference" would be more precisely replaced by "length of the vertex" by not the total extent, but only over the forehead, apex and occiput extending to the front edge of the medullary hole Arc, thus reducing the reduced around the skull base vertical circumference in the table. As already in III. Chapter notes, the measurements were provided by Prof. WELCKER, who has collected a rich, evenly treated material while adhering to one and the same measurement method. [1] The unit of measurement is the millimeter. A tape measure was used for the measurement. The measurements themselves refer to WELCKER's statement on "normal" male skulls. Skull with suture abnormalities, even frontal skulls were excluded.]

[1] [Comp. H. WELCKER, Growth and Construction of the Human Skull, Leipzig 1862; furthermore: The capacity and the three major diameters of the skullcap at the different nations; Archive for Anthropology, Vol. XVI].

[Plate III contains the recruiting measures of 2,047 20-year-old Leipzig students from the 20 years 1843-1862. From the original list of these measurements, it is to be noted that it is characterized by a pure randomness in the series of measures established in its method of production the same in Chap. XX is used to prove the extreme laws. The unit of measurement is the Saxon customs = 23.6 mm; however, not only the whole but also half and quarter inches were measured.]

[Plate IV shows the dimensions of the uppermost member (internode) of 217 six-membered rye stalks. More detailed information on the extraction of this material can be found in the second part, chap. XXV. It is connected with the measuring method just described that half a centimeter appears as a unit of measure.]

§ 50. [The four panels are in order: [2]]

Plate I. 450 europ. Men's skull; Vertical circumference .

$E = 1$ mm; $m = \sum z = 450$; $A_1 = 408.5$.

a	*z*		*a*	*z*		*a*	*z*
368	1		400	13		425	8th
371	2		401	12		426	7
376	1		402	13		427	3
378	1		403	6		428	4
379	1		404	10		430	3
380	2		405	18		431	3
381	1		406	8th		432	2
382	2		407	8th		433	5
383	3		408	16		434	5
384	3		409	13		435	4
385	8th		410	20		438	1
386	2		411	9		440	3
387	6		412	15		442	1
388	4		413	8th		443	1
389	5		414	12		447	1
390	7		415	21		448	1
391	7		416	6			
392	7		417	5			
393	2		418	16			
394	8th		419	9			
395	12		420	15			
396	4		421	8th			
397	7		422	7			
398	14		423	5			
399	3		424	12			

2) [Since neither the original lists nor the primary panels of the K.-G. (see note to Chapter III), the above panels had to be reconstructed. Panel I and III could from the five resp. four reduction layers, which are listed in the following chapter (§ 64 and 65), are restored; for panels II and IV the corresponding arrangements were not sufficiently complete. In the meantime, the logarithms of *a*-Values. The values of Table II, on the other hand, were derived from the measurements of 500 European male skulls transmitted to me by Prof. WELCKER. However, 63 measures had to be added according to their probable affiliation to the corresponding vertical dimensions, since only in this way could a match be achieved with the reduced table of the following chapter (§ 58). However, the slight deviations that may result from this do not affect the image of the panel, which, moreover, is not materially considered below.]

Panel II. 450 europ. Men's skull; Horizontal extent.

$\boldsymbol{E}$ = 1 mm; $m = \sum z_{,} = 450$; $A_1 = 522.2$.

a	*z*		*a*	*z*		*a*	*z*
481	1		510	13		535	10
484	2		511	12		536	11
485	2		512	14		537	5
486	1		513	7		538	8th
488	1		514	6		539	9
489	2		515	13		540	14
490	2		516	11		541	6
491	1		517	7		542	3
492	1		518	9		543	4
493	2		519	10		544	3
494	4		520	15		545	4
495	5		521	6		546	3
496	1		522	8th		547	2
497	4		523	14		548	2
498	1		524	17		549	3
499	2		525	2 1		550	6
500	8th		526	9		552	1
501	4		527	8th		553	1
502	3		528	7		554	4
503	6		529	8th		555	2

504	9		530	13		558	1
505	8th		531	5		561	1
506	4		532	6		567	2
507	3		533	7		576	1
508	6		534	8th			
509	7						

Plate III. Student recruit measurements .

E = 1 inch, $m = \sum z = 2047$; $A_1 = 71.77$.

a	***z***		***a***	***z***		***a***	***z***
60,00	1		70,00	70		76,00	24
64,00	2		70.25	65		76.25	17
64,75	4		70.50	71		76.50	9
65,00	6		70.75	61		76.75	7
65,25	3		71.00	78		77,00	14
65,50	5		71.25	75		77.25	9
65.75	5		71.50	81		77,50	7
66,00	8th		71.75	89		77.75	3
66.25	6		72,00	79		78,00	3
66,50	9		72,25	81		78.25	2
66.75	19		72,50	82		78,50	3
67,00	7		72.75	63		79,00	1
67.25	11		73,00	79		79,50	2
67,50	25		73.25	79		80,00	1
67.75	15		73,50	68		80.75	1
68,00	35		73,75	56		82,50	1
68.25	27		74,00	64			
68,50	37		74.25	42			
68.75	34		74.50	55			
69,00	43		74.75	33			
69.25	48		75.00	43			
69,50	57		75.25	26			
69.75	54		75,50	25			

			75.75	17			

Plate IV. The uppermost member of 217 six-membered rye stalks.

$\boldsymbol{E} = 0.5$ cm; $m = \sum z = 217$; $A_1 = 86.54$.

a	*z*		*a*	*z*		*a*	*z*		*a*	*z*		*a*	*z*
42.9	1		75.6	1		85.4	1		91.7	1		99.0	2
49.7	1		75.8	2		85.5	1		91.9	2		99.2	1
52.8	1		76.1	1		85.7	1		92.0	2		99.3	1
55.6	1		76.2	2		85.8	1		92.3	1		99.4	1
57.6	1		76.4	2		85.9	1		92.8	1		99.5	1
58.9	1		76.7	1		86.0	2		93.0	2		100.3	1
59.0	1		77.0	1		86.2	1		93.1	1		100.5	1
61.4	1		77.2	1		86.3	1		93.3	1		100.8	1
61.9	1		77.5	1		86.8	2		93.4	1		100; 9	1
62.2	1		77.6	1		86.9	1		93.5	2		101.0	1
62.3	1		77.7	1		87.0	3		93.7	1		101.1	1
63.0	1		77.9	1		87.1	2		94.4	1		101.3	1
64.1	1		78.0	1		87.4	2		94.6	2		101.5	1
64.3	1		78.1	2		87.5	1		94.7	1		101.9	1
65.5	1		78.4	1		87.8	1		95.7	1		102.2	1
67.4	1		78.8	1		87.9	2		95.8	2		102.3	1
67.7	1		79.0	1		88.0	2		95.9	1		102.7	1
67.8	1		79.4	1		88.3	1		96.0	1		102.8	1
68.1	1		80.0	2		88.6	1		96.1	1		103.3	1
68.3	1		80.4	1		88.8	1		96.2	1		103.4	1
68.9	1		80.7	1		88.9	2		96.3	1		104.0	1
69.6	1		80.9	2		89.2	2		96.5	1		104.2	1
69.9	1		81.3	1		89.3	2		96.8	1		104.4	1
70.5	1		81.9	1		89.4	1		96.9	1		105.3	1

71.4	1		82.0	2		89.7	2		97.0	1		105.5	1
72.0	2		82.1	2		89.9	2		97.1	1		105.6	1
72.1	1		82.3	3		90.0	1		97.5	2		105.8	1
72.5	1		82.4	1		90.2	3		97.6	1		106.0	1
72.9	1		82.8	1		90.4	1		97.7	1		106.2	1
73.7	1		83.0	1		90.5	1		97.8	1		106.3	1
73.9	1		83.1	1		90.6	1		97.9	1		108.0	1
74.1	1		83.4	1		90.7	3		98.0	1		110.0	1
74.8	2		83.7	4		91.2	1		98.2	1		111.2	1
75.1	2		83.9	2		91.3	1		98.6	1		112.0	1
75.2	1		84.6	1		91.4	1		98.8	1		112.2	1

§ 51. [A comparative look at these tables shows, as regards the course of the *z* as well as the juxtaposition of *a,* a substantial difference of the first three tables from the last. The former possess in fact a central main constituent, the *z* of which grow generally towards the center of the table, and whose *a* , apart from individual interruptions towards the ends, form an equidistant series. Thus in I the equidistants *a* extend continuously from 378 to 428 and from 430 to 435, while at the same time the *z*, but with constantly recurring fluctuations, first grow and then decrease again. In II., The series of equidistants *a* goes from 488 to 550 and, after interruption by the missing *a* = 551, continues from 552 to 555, while again the *z* *show*a similar course. Plate III. finally, with a corresponding behavior, the *z* between the limits 64, 75 and 78, 50 is characterized by an undisturbed equidistance of *a* . This main inventory is followed in each of the three panels at the beginning and end by a relatively small number of *a* values whose distances change randomly, and whose *z*predominantly equal to 1: they represent end divisions with scattered *a* . In the fourth panel, on the other hand, the *a go* consistently at irregular intervals, and it can only be remarked that the smaller intervals occur more frequently in the middle than at the ends; At the same time, the vast majority of *z is* equal to 1. Thus, we can have plates that have a main stock of equidistant *a* next to end divisions with scattered *a* , and those whose *a* through the whole panel through irregularly dissipating, differ. As representatives of the first type, the plates I. to III. to apply; Plate IV represents the second type. Both types are essentially different from each other; for it will be shown that tablets of the second type require a much greater reduction than those of the first, if their treatment is to succeed.]

[When delimiting the main stock of a blackboard, however, it must be taken into account that it does not detach itself from the final departments in sharp definiteness. It would be possible to counteract any indeterminacy by setting up the cones, that the main constituent should extend as far as the aquidistance of the *a* . However, it is clear from the outset that no essential provision would be made. For in many cases the case may arise that even against the middle of the tablet the equidistance is disturbed by a missing *a* ; even more frequently, from the middle towards the beginning or towards the end *, there will be* another row of equidistants *a* , as is indeed the case for I and II, owing to the absence of*a* = 429 resp. *a* = 551 is true. In such cases, if the above rule were adhered to, the main inventory would either be unduly limited or totally questioned. On the other hand, it is also possible that the *a* may be complete, but the course of the *z* makes their exclusion from the main stock seem desirable. It must therefore be left to the determination of the main stock within a certain latitude of arbitrariness, since a rule can be set up only to the extent that the equidistance of the *a* values are not subject to considerable disturbances and with respect to the *z,*at least on the whole, a growth towards the center should be recognizable. Thus we can state the limits of the main stock for I 378 and 435, for II 488 and 555, for III 64.75 and 78.50, with the remark, however, that these limits permit a shift.]

[Incidentally, the equidistance of *a can be* at least formally established even in the case of missing *a* , if the missing *a* , with a *z* = 0, are included in the panel. It should be referred to as insertion of empty *a* . For example, the main population of I and II in this way becomes consistently equidistant when inserted in I 429, in II 551 with a z = 0.]

As for the course of *z* in the main section of plates I-III, it has already been remarked that the increase towards the middle is subject to constant fluctuations. Now, however, uninterrupted growth and decline is not to be expected because of the never-missing unbalanced contingencies. But if this alone is the cause, the unmistakable periodicity in the wavering of the *z* would remain inexplicable. There must therefore be another cause behind it. The same is evident from the following remarks.]

[In the main constituent of I there are altogether 18 relative maxima, 17 minima lying between them; 8 maxima fall on those *a,* which represent whole or half centimeters, while no single minimum belongs to such *a* . Of the 17 maxima of the main population of II, 10 fall from the 16 minima, none to *a*This shows quite well that in the measurement of the skull by means of the tape measure, apparently the millimeters were obtained by estimation, whole and half centimeters were preferred; otherwise the maximums and minima would have to be distributed equally among the subdivisions of the centimeter. In the non-uniform estimation, ie in the preference of the whole and half divisions of the scale used, one finds thus the source of periodically recurring irregularities in the course of *z.* This is confirmed on Plate III. Of the 19 maxima of their main stock, 9 fall to full, 7 to half inches; of the 18 minima, only 2 integer values are added, while the remaining ones are ¼ or ¾ in.

[It will therefore be necessary to guard against errors due to non-uniform estimation in the processing of the distribution panels and to consider their elimination by an appropriate reduction. This results in dividing the panels, according to the period of non-uniform estimation, into main divisions. For example, in panels I and II, they must progress from 5 to 5 mm, in panel III, after half an inch, or better, by whole inches. In general, these main sections will be started with the main section of the board. One may then find it advantageous to circumscribe the main stock so that it holds just a full number of principal departments. Then z. For example, in Table I, three values are truncated from the inventory as defined above and, for example, values 380 and 434 are selected as boundaries,

§ 52. [Finally, the following points valid for each distribution panel in their entirety must be mentioned. Each measurement is subject to limits of accuracy so that the *a can* never string together continuously but must be separated by an interval whose magnitude depends on the degree of accuracy of the measurement. This interval should be called the primary interval and be denoted by *i* . It is constant for the extension of the whole table, since it is conditioned only by the scale, not by the size of the measured objects.]

[In its existence one has to look for the reason that an equidistant main stock in the distribution boards is even possible. For the interval of the main stock is nothing else than the primary *i,* which can not be undercut, but only the more clearly the larger the number of measured copies of the K.-G. - the *m* the board - is. Of course, the primary *i* can also be seen directly from the *a* values for boards without a main stock. For blackboard IV z. For example, it is equal to the tenth part of ***E*** , di = 0.05 cm.]

[The essential meaning of the existence of a primary interval, however, consists in the fact that it places in the correct light the belonging of the *z* to the *a,* which are added to the tables. For it is evident that the *a* are to be understood merely as representatives of the primary intervals whose centers they represent; It is therefore also the *z* rather than the *a,* but as the by *a* designated conceiving associated primary intervals and think distributed within the latter uniform because it lacks at each stop for a differently designed, legitimate distribution. Insofar as the primary interval is the *a*encloses or circling, it should be called the perimeter interval of the *a* . Its mutual boundaries are *a* - ½ *i* and *a* + ½ *i* ; These are joined together by the whole table, so that the first boundary of any one interval coincides with the second of the preceding one.]

[The *a* and *z* values are thus bound together by the associated perimeter interval. If this connection is to be dissolved, and the *a* is considered and understood by itself, then it shall be called bare *a* .]

[The affiliation of the *z* to the *a* explained just now allows a true geometric representation of the distribution tables. Namely, the *a* in an abscissa line are to be applied and by marking the values *a* - ½ *i* and *a* + ½ *i* to add the perimeter intervals of the same; then are to be established on the latter rectangles, the content of which the *a* beige signed the panel *z* must represent; Of course, both the dimension of the *a* , as well as the construction of the rectangles are based on any scale, since it is only necessary to gain a picture of the proportions of the Tafelwerte.]

[You get such z. B. the following representation of the middle part of Table I:]

Fig. 1.

VIII. Reduced distribution boards.

§ 53. Partly to make the distribution boards more confined and thus to occupy a smaller space for them, partly to make up for the irregularities in the course of their values and to make harmless any non-uniformities of the estimation, partly the calculation of the determinants or so-called elements of the K.-G. In order to facilitate, one has to go from the primary distribution tables to the reduced ones and let them enter for those, and, notwithstanding certain relations, a primary table can not be replaced by a reduced whole, in fact the reduced table retains advantages over the primary in given relations and it becomes necessary to deal with their composition, their conditions and their method of utilization.

Let us first consider the reduction of such primary plates, which, like I to III, make one main constituent with equidistant *a distinguish* from end divisions with scattered *a* . In order to produce a reduced one from a primary plate of this kind, we divide, as has already been mentioned in § 50, the main component of the same in compartments, which in their *a*- column have an equal number of equidistant ones (if necessary *,* made equidistant by insertion of a blank) , so called bare *a* , and in particular sums the *z of* each of these sections. After this applies as a reduced *i*the size of the entire interval in which the number of primary *a* is, including its radius intervals summarized as a reduced *for* the sum of *z,* which on the information contained in the reduced intervals *a* fall, as a reduced *a,* wherein the reduced *z* is beizuschreiben , the means between the whole bare *a* or, what comes out the same, the means of the extreme naked *a,* which enter into the interval.

By way of explanation serve the reduction of a particular section of the main inventory of the primary panel I as:

naked *a*	380	381	382	383	384
primary *z*	2	1	2	3	3

By summing the primary *z* , we obtain as reduced *z* the number 11, while the reduced *a obtains* the mean from the 5 primary bare *a of* the division in question or, because of its equidistance, the mean from the outermost *a* , 380 and 384, hence 382 is to which the reduced *z* = 11 is attributed. The limits of the reduced *i,* however, are not the outermost naked *a*380 and 384, and therefore the reduced interval is not 384 - 380 = 4, because in the reduced interval also the perimeter intervals of the boundary *a*with which the whole interval extends to one and the other side by a primary ½ *i* ; since the primary *i* = 1, the limits of the reduced interval to one side are 380 - ½ = 379.5, after the other 384 + ½ = 384.5, and the size of the whole reduced interval is the difference of both = 5 ,

Thus, while one obtains the reduced *a* itself as a means of the outermost primary naked *a* , which enters into the department to be reduced, the size of the reduced interval can not be obtained as the distance between both boundary *a* , but only by extending this distance to each side for a half, therefore, on the whole, a whole primary *i.* This is probably to be considered and not everywhere right, as further noted.

If *n* equidistant nude *a* and thus *ni* are united in each main section of the primary board, then the *i of* the reduced board is also *n* times the *i of* the primary board. Now, in each of the main sections of Tables I and II, there are 5 each, and in III 4 naked *a* in each main section; the primary *i* at I and II is 1 mm, at III ¼ inch; that is, *i of* the reduced plates at I and II equal to 5 mm, at III equal to 1 inch.

§ 54. As in the case of the primary panels, one does not assume that the reduced *a* itself occurs so often as the reduced *z* ascribed to it, but that the interval represented by the reduced*a* , *z* Divide values *a* , which keep between the limits of the reduced interval; and so long as the *a of* the primary plates represent, as a matter of principle, a whole interval to which their *z is* distributed, only a smaller one than the reduced *a, it* is basically between primary and reduced *a*only a relative difference. Instead of the reduced *a,* however, the interval itself can also be given in the reduced tables, which is represented by it, and one and the other is present in the reduced tables so far, according to which I distinguish *a*- plates and interval-tables. Only because of the somewhat shorter presentation, I usually prefer the shape of the *a*- board; but a factual distinction does not exist between *a*- plates and interval-tables, and one can easily come from one form to the other, provided that the reduced *a of* the *a*Table is the mean between the limits of the reduced intervals, while the boundaries of the intervals are the same as in the primary tables *a* - ½ *i, a* + ½ *i* , except that here reduced *a* and *i take* the place of the primary as in the following examples, in which the reduction according to the stated principle is continued by a division, and the following corresponding *a*- column and interval column are hereby obtained:

Red. ***A***	**red. intervals**
382	379.5 - 384.5
387	384.5 - 389.5

If, in our example, we continue the reduction according to the same principles through Table I, we obtain the following reduced *a*- and interval- table belonging to each other :

a	**intervals**	***z***
382	379.5 - 384.5	11
387	384.5 - 389.5	25
392	389.5 - 394.5	31
397	394.5 - 399.5	40
402	399.5 - 404.5	54
407	404.5 - 409.5	63
412	409.5 - 414.5	64
417	414.5 - 419.5	57
422	419.5 - 424.5	47
427	424.5 - 429.5	22
432	429.5 - 434.5	18

It can be seen in this example that the intervals of the reduced panel coincide with each other by coinciding the second boundary of each interval with the first boundary of the following interval, and the respective interval boundaries of the primary panels (see § 52).

Not everywhere but can be found elsewhere, the interval boundaries after previous rule specified correctly, but neglecting the radius intervals which border *a* self given the reduced departments as interval limits, in the otherwise estimable Belgian Rekrutenmaßtafeln, but this appears justified insofar as the experience immediately but only this border *a* are, from where you can easily pass on the true interval limits for recycling of the panels; but it would seem more advisable to give the true borders, even according to the previous rule, in the tables. If the designation of the interval limits in the sense of the Belgian tables in our tables is to be done, we would in our previous example, the *a*-Connect panel to the interval table, set:

a	I **ntervalle**	*z*
382	380 - 384	11
387	385 - 389	25
392	390 - 394	31

etc

But the disadvantage of this notion is that the intervals do not close, but leave gaps of one unit of measurement between them, into which in reality there are dimensions of which the table is not accountable. However, one raises this evil and can lift it even in the Belgian tables of measurement by making these limits coincidental by drawing from the limits of successive intervals.

§ 55. What we have outlined above with an example of the skulls will be applied to all tablets which have a main stock with equidistant *a* . But if we apply this application to the Student Measurements Panel III, an inconvenience occurs, which can be countered by a procedure which I shall call the reduction with divided *z* . For explanation, let us refer to the first two divisions of the main stock of Primary Panel III. You are:

Naked *a*	65.0	65,25	65.5	65.75	66.0	66.25	66.5	66.75
Primary *z*	6	3	5	5	8th	6	9	19

Where i = 0, 25 inches.

If we now reduce these divisions to four times the primary i according to the previous rules, we obtain the following *a* and interval tables, which are afflicted with extremely inconvenient fractions :

reduced

a	**intervals**	*z*
65.375	64,875 - 65,875	19
66.375	65,875 - 66,875	42

In fact, the reduced *a* = 65.375 is the average of the primary limit *a* 65 and 65.75 and the reduced interval limits 64.875 and 65.875 are equal to the reduced *a* = 65.375 ± half the reduced *i.*

[To counter this inconvenience, note that the main constituent of a chalkboard with equidistant *a* is not in sharp distinction from the end divisions with scattered *a* . One might III instead of 65.0 the main constituent of the table just as much empty with 64,75 or after insertion of *a* can, start with 64.5 or 64.25. However, such a shift of the main population by one, two or three whole primary *i* would not lead to the goal; for even after the shift, both the reduced *a* and the bounds of the reduced intervals would be in the middle between any two adjacent primary *a*fall and still be afflicted with the uncomfortable fractures. Note, therefore, further, that, as has already been remarked several times, the *z of* the table is not directly listened to in the letter *a* , but is distributed over the entire perimeter interval of the *a* . It is thus permissible to divide the primary *i and* to divide proportional portions of the *z* into the subintervals . In particular, one can halve the primary *i* , so that instead of the interval with the boundaries *a* - ½ *i, a* + ½ *i* two intervals with the limits *a* - ½ *i, a* and *a, a* +½ *i* , each of whom ½ *z is* listening. If the latter occurs in the primary panel III, for example, instead of:

primary

intervals	*z*
64,875 - 65,125	6
65.125 - 65.375	3
65.375 - 65.625	5

etc

the following associated interval and *z* series:

primary (halved)

intervals	*z*
64.875 - 65.0	3
65.0 - 65.125	3
65.125 - 65.25	1.5
65.25 - 65.375	1.5
65.375 - 65.5	2.5
65.5 - 65.625	2.5

etc

If one now shifts the main stock by half a primary *i* instead of a whole *,* and starts it with 65,0 instead of 64,875, which values mean interval limits and not *a* values, the following *a* and interval table are obtained :

reduced

a	**intervals**	***Z***
65.5	65.0 - 66.0	20
66.5	66.0 - 67.0	41.5

However, if one starts the main stock with 64.5 as the interval boundary, one obtains:

reduced

a	**intervals**	*z*

65.0	64.5 - 65.5	15.5
66.0	65.5 - 66.5	26

In this way, by shifting and dividing the intervals, it can always be achieved that at least the interval limits or the *a* values of the reduced slab become integral if only the reduced *i* *is*equal to or a multiple of the underlying unit of measure.]

§ 56. Now, however, there are also tables, such as Plate IV for the rye ears, where the dimensions are very scattered throughout the table, where a main stock with equidistant *a* is absent from the outset and only by a virtually barely feasible interposition of countless empty *a* could be produced. Then you will have to proceed as follows.

First, one has to be drawn up by the immediately (§ 60) considerations to decide on one as large *i* you want to reduce. In order to obtain a nearly regular progression of the values *z,*one will not be allowed to go under four units of measure on our board with *i* . Let us now proceed to include the first primary *a* = 42.9 in the first reduced interval, with its first limit so far back that this purpose is achieved, to which suffice the first limit of the first red. Interval = 42, then 42.9 falls within the first interval 42-46 [1)]. The reduced *z of* this interval is then the sum of the primary *z*which fall into the interval 42-46, ie 1, the red. *A* the middle between 42 and 46, thus 44. The second red. Interval is hereafter 46 - 50, where again only one *z*falls, hence the red. *z* = 1, etc., which gives the following reduced table from the outset:

reduced

a	**intervals**	***z***
44	42 - 46	1
48	46 - 50	1
52	50 - 54	1
56	54 - 58	2

If one of the interval boundaries randomly with *a* coincident the primary panel, the half primary is *such* that *a* in the reduced *z* of the interval to take by the other half *for* (such as by the method of the divided *z*) belonging to the neighboring interval.

[1)] For the same purpose one could go back even further with the first limit, to 41, to 40, to 39, where then the first intervals would be 41-45, 40-44, 39-43. In each of them fell 42,9. This gives different layers of reduction, of which afterwards; but in any case 42 is sufficient as the first interval limit for the purpose.

§ 57. If we now return to distribution tables such as I, II, III, in which a main constituent with equidistant *a of* the *a*- column can be distinguished from end-divisions with scattered *a* , it is still necessary to specify how to proceed with the latter. This can be done in two ways. Either α) make the *a of* the final divisions just as equidistant by the insertion of empty *a* , as is the case in the principal departments, and reduce them according to previous principles, since in principle they no longer differ from the principal divisions; or β) one does not continue the reduction by the end departments, but is satisfied with Bausch information about it. The latter method, as far as I can see, has hitherto been the only customary procedure, but the former method is preferable to the reasons to be given, and in the future alone.

So you see everywhere after process β) in the case of recruits, precede the reduced main stock by the bulge of the number of measures smaller than the first limit of the reduced main stock, and close the bulge table with the number of measures greater than the second limit of the reduced principal stock, without specification of these dimensions, which one should not limit oneself to, since one then still the central value, but can no longer determine the arithmetic mean, not to commemorate other disadvantages; rather, if one wishes to renounce the reduction by the end divisions at all, then, besides the sum of the number of measures, the sum of the measures themselves, which are contained in the end divisions, should be given, and it is not inadvisable to add the primary extremes ,*v* and Vorsumme *V* , the number ($\sum z$) and sum ($\sum az$) of the primary *a,* which are smaller than the first boundary of the reduced main stand, on the other hand as Nachzahl *n* and Nachsumme *N* the number and sum of the primary *a,* which is greater than the The second boundary of this population, as $E_{\prime}$ and *E 'is* the smallest and largest *a of* the whole primary table, is the reduced main population by giving *v* , *V* , *n* , *N* ,$E_{\prime}$, *E '* supplement, which one makes useful the table, but of course it to the advantage of brevity, the only pure β , loses procedure granted.

But the method α) is not only more methodical, since afterwards the reduction of the whole primary panel can be carried out without the always arbitrary demarcation between main stock and end departments and without a supplement of the last kind according to the same principle, but strictly speaking only so reduced panels are useful for the distribution calculations to be made.

I lead now on this principle, the reduction to a $i = 5$ mm through all the tables I and II, with respect, by switching empty *a* not only *a* make around the table equidistant, but also the first primary force *a* so many empty *a* let precede that the first primary *a* still falls into it (with I 368, wherein II 481) in said first reduced interval, we can empty to fulfill this condition, depending on the selected reduction layer 1, 2, 3 or 4 *a* preceding Let, for example, if one precedes two, the first be replaced by empty *a*supplemented departments of the primary panel I have to write:

primary *a*	366	367	368	369	370	371	372	373	374	375
primary *z*	0	0	1	0	0	2	0	0	0	0

etc

The first red. Interval is hereafter, with respect to the perimeter intervals of the primary boundary *a* , 366 - ½ to 370 + ½, di 365½ - 370½, the second 370½ - 375½; the red. *a of* the first interval is 368 as the middle between 366 and 370, the second 373; and the reduced *z* obtained by summing the primary *z of* each section is for the first section 1, for the second 2, which gives as the beginning of the reduced board:

reduced

a	**intervals**	*z*
368	365.5 - 370.5	1
373	370.5 - 375.5	2

etc

Accordingly, in Plate II we shall have to write the first two sections supplemented by empty *a* :

primary *a*	480	481	482	483	484	485	486	487	488	489
primary *z*	0	1	0	0	2	2	1	0	1	2

Hereinafter as the beginning of the reduced board:

reduced

a	**intervals**	*z*
482	479.5 - 484.5	3
487	484.5 - 489.5	6

§ 58. If we introduce now this reduction through all the tables I and II, we get under restriction to the shape of *a* following reduced panels, each a very useful for future column -self- *S* , is attached, which is created by in that the *z* of the *z* column from the beginning to the beginning *a* (incl.) of *a* column, to which the respective *S* , is joined summed:

Reduction of the primary panels I (vertical perimeter) and II (horizontal perimeter) with red. *i* = 5 mm.

I II

a	*z*	*S* ,		*a*	*z*	*S* ,
368	1	1		482	3	3
373	2	3		487	6	9
378	5	8th		492	10	19
383	17	25		497	13	32
388	24	49		502	30	62

393	36	85		507	28	90
398	41	126		512	52	142
403	59	185		517	50	192
408	65	250		522	60	252
413	65	315		527	53	305
418	51	366		532	39	344
423	40	406		537	43	387
428	17	423		542	30	417
433	19	442		547	14	431
438	4	446		552	12	443
443	2	448		557	3	446
448	2	450		562	1	447
				567	2	449
				572	0	449
				577	1	450

The comparison of the foregoing reduced tables with the primary ones from which they are based gives rise to the following remarks of general significance.

I understand at all under a regular course of *z* such that they grow with ascending *a* without interruption by descending to a maximum, but from then on decrease equally without interruption by ascending, thus giving a smooth distribution curve in the sense of § 17, Thus, at first glance, all reduced panels show the most striking advantage of regularity against the primary ones from which they are derived. And only after the course of values through the reduction has become regular at least around the middle, will it speak of a legality of it, determine it, or have it examine a presupposable legality.

That I two adjacent same maximum *z* shows just happens and is not the regular way in the way, as would be the case if by intervening *a* smaller *z* were divorced. II has, as usual, only a maximum *z* . Closer to attention, I shows an insignificant exception to regularity only at one end, as long *as z* = 17 and 19 have to interchange their size in order to follow properly; and there are seldom any such small irregularities towards the ends, without much importance being attached to the use of the tablets, the more so if those in the region of the densest *a,*ie, the largest, *for example, are* the most importanthas take place; and if, for the sake of brevity, we understand by the kernel of the tablet the densest *a* with its two higher and two lower neighbors *a* , then we shall preferably have to require regularity from this kernel in order to find our normal laws of distribution confirmed with satisfactory approximation. Now, while the kernel of I, which extends to six *a* because of the double maximum *z* , satisfies the condition of regularity, this is not the case with respect to II upwards (according to the smaller measures), and also to the lower order the number 43 incorrect against the limit number 39 of the core.

From this it can be inferred from the beginning that panel II for horizontal circumference will be less suited to the normal mode of distribution and will be less suitable for proving normal laws than panel I for vertical perimeter.

§ 59. But now it suffices Plate I and II to twice *i* than before, rather than reduced to 5mm to 10mm in order to make the two tables, invariably regularly, which can be done by very simple, that two successively *a* of on *i* = 5 mm reduced tablets to their mean and their associated *z combined* to the sum. If this is done from the top with the tablet I (§ 58), then because of the unpaired number of the naked *a of* this table, the *a* = 448 with *z* = 2 remains; but it does not hinder consistency to continue the *a*- board beyond 448 by adding a 5mm larger to the *a* = 448*a* = 453 with *z* = 0 adds; the mean *a* between 448 and 453 then gives a reduced *a* = 450.5 with a reduced *z* = 2. In fact, the following tables are obtained:

Sheets I and II, reduced to *i* = 10mm.

I II

a	***z***	***S*** ,		***a***	***z***	***S*** ,
370.5	3	3		484.5	9	9
380.5	22	25		494.5	23	32
390.5	60	85		504.5	58	90
400.5	100	185		514.5	102	192
410.5	130	315		524.5	113	305
420.5	91	406		534.5	82	387
430.5	36	442		544.5	44	431
440.5	6	448		554.5	15	446
450.5	2	450		564.5	3	449
				574.5	1	450

From the previous panels one will, on the same principle, be able to derive a table reduced to *i* = 20 mm, and so on, which I call different reduction stages. With each new reduction stage, the board shrinks, until at last one single red. *a* with a single red. *z* comes.

In order to carry this out only for panel I, the following *a*- plates are obtained for reduction, respectively, to 20, 40 mm, and so forth from the reduction for *i* = 5 mm :

20 mm 40 mm 80 mm 160 mm

a	***z***		***a***	***z***		***a***	***z***		***a***	***z***

375.5	25		385.5	185		405.5	448		445.5	450
395.5	160		425.5	263		485.5	2			
415.5	221		465.5	2						
435.5	42									
455.5	2									

And so, if a reduction of a given *i is* still not possible to obtain a regular course of the values *z* , we will be able to arrive at such a value or at least be able to approach it by enlarging the *i* . And, as it is easy to understand, the possibility of reduction to a different size *i* exists right from the start . We could have at I and II, the primary *i* at the first reduction stage by more or less than five times, by more or less than four times in III *i* can increase as we more or less equidistant (resp. By inserting empty *a* equidistant made) primary *a*taken together. So these are aspects that can determine the choice in this regard. Quite general and firm for each special case presented can not be well now, but the following set up, which can restrict and regulate the freedom of choice to a certain extent.

§ 60. There is a certain conflict between the advantages and disadvantages of enlarging or reducing the reduction *i* . From some points of view, it is most advantageous to keep the *i* as small as possible, because, according to earlier (§ 5) plodded discussion, the ideal laws of distribution strictly presuppose this case, and in this respect even the primary table deserves priority over any one that reduces always contains a multiple of the primary *i* ; yes, it would be best if the *i of* the primary board itself could be reduced to infinite smallness, which of course is not possible. The following circumstance also contributes, under otherwise similar circumstances, the reduction to small *i* to let the reduction prefer to larger ones. Let the fact that the number *z* written on a given *a* really belongs to a whole interval, which in the case of primary and reduced plates of the size of *i*growing, to be taken into account when determining the elements, what must be done later (chapter IX), interpolation of the interval in question, and it may be necessary to keep the intervals small enough to suffice with simple interpolation; for the collective theory of measure would become practically almost impracticable if one had to draw interpolation with second differences everywhere in order to determine all elements and the comparisons between calculation and observation. And although I will give the procedure for this later, I have generally not made use of it, since, limiting myself to the magnitudes of the *i, I* have not been able to derive any advantage that would render the uselessness of the use and circumstantial representation invalid.

The contrary, the adjustment of the contingencies that the regular course of *such* interfering in the primary panel and are the comparisons with the legally required progress in the way, but only by reduction and hereby increasing the *i* are obtained, and a not too big *i* hurt much less than too much irregularity in this respect. Hereinafter, it will do the most as a whole, the *i* as large and yet not to be taken greater than that a regular transition occurs at least within the core of the reduced table; because irregularities in the course of the outermost small *z* have no significant disturbing influence on the determination of the elements and legal conditions. But where, as in the case of our first three examples, irregularities due to unbalanced contingencies still occur because of non-uniform estimation, there is the special condition that *i* should not be smaller, and consequently the number of equidistants *a to* be summed up is not less the period of the non-uniform estimation, and if the *i is* magnified, this is done only after a whole multiplis, because only under this condition is it possible to compensate for the errors due to non-uniform estimation. Now, in the case of the skull dimensions of Tables I and II, according to § 51, the maximum dimensions*z* after each 5 mm advancing by 1 mm *,* in the student recruitment measures of Tab. III after 4 each by 0.25 inch advancing *a of* the primary board again, so the reduction to the smallest stationary *i* in I and II can only be to *i* = 5mm , in III only 1 inch, as is the case in the tables (§ 58 and § 62); but to respond to a larger *i* would only be a cause if it would not be possible to achieve a regular course of the reduced z.

§ 61. Although, for reasons given, there will be no reason to proceed to these higher stages of reduction in the elaboration of our panels of examples, it may nevertheless have an interest to see in them how far such progress can be expected to bring about a change in the elements. and I give first of all, for Table I, the following table of the most important elements, depending on their derivation from different stages of reduction. The determination of D_p has been done only for the first two stages of reduction because of its complexity. After changing the main values, of course, also the dependent deviation functions change; *u* , ***u*** and *p*previously (§ § 10 and 33) have the meaning given, from which μ', $\mu_,$, ***m*** ', ***m*** , with the concurrence of the total number *m* can be concluded in the manner indicated. The derivative of ***m*** ', ***m*** , and thus of ***u*** , as well as of *e'* , *e* , has been done everywhere from D_p, not from D_i. The *A* derived from the primary panel , ie A_1, is mentioned in the title. All elements are derived according to the so-called sharp method of Kap.IX and X with simple interpolation of the intervention interval. Correspondingly, all further panels of the elements are to be understood.

Elements of Table I, depending on the derivation of different reduction stages .

E = 1mm; *m* = 450; A_1 = 408.5.

i	5 ***E***	10 ***E***	20 ***E***	40 ***E***
A_2	408.2	408.1	408.2	409.2
C_2	408.6	408.6 [2)]	409.1	411.6
D_p	409.7	410.1	-	-
D_i	410.5	409.8	410.6	414.7
u	+ 10	+12	+ 20	+31
u	- 29	- 40	-	-

e ,	11.9	12.4	-	-
e '	10.4	10.4	-	-
p	0.74	0.75	-	-

[2] It might appear as an oversight that C_{2} has been given the same value for i = 10 as for i = 5. However, this is due to the fact that the interval in which C_2 falls for i = 10 has z twice as large as the interval into which C_2 falls for i = 5, which is represented by the two neighboring equal maximum z the reduction level i = 5 is conditional.]

It can be seen that, apart from the last reduction step taken into account here, at i = 40, where the reduced board shrinks to three values, the principal values deviate only by negligible and seemingly random magnitudes, depending on the reduction stage; whereas u , $\boldsymbol{u}$, and hence μ , , μ', $\boldsymbol{m}$, , $\boldsymbol{m}'$ to change not insignificantly thereafter, from which it may be concluded that, if it is only a matter of determining the principal values, the reduction stage does not matter much, if only one does not go to the highest levels; whereas the distributional computations of the reduction stages must be essentially influenced, and it is therefore also for this reason that it will probably do, as far as it is concerned, to compare with calculated distribution, stand at the lowest possible level, which still gives a regular distribution in the nucleus stay. Now, where the lowest degree is not due to a non-uniform estimate, as in Plates I, II, and III, one is not bound either, the first chosen $i$just to double in order to arrive at the purpose of a regular nucleus, which has only the formal advantage that one can simply derive the higher degree from the previous lower level. But if one can obtain a regular kernel by means of a weaker reduction than by doubling the previous i , one will not resort to this doubling, but must then go back to the derivation of the respective reduction on the primary panel.

§ 62. Now to see how these results compare with other K.-G. In other circumstances, we turn from Plate I, which applies to skull measures with m = 450 [3] , to Plate III for student recruitment measures with m = 2047.

[3] I pass over Panel II, not only because it presents analogous conditions as I, but also because it offers less certainty in the kernel of the primary panel because of irregularity.

In panel I we were forced by the behavior of the non-uniform estimate to reduce the primary i = 1 mm in the first stage to five times; in Plate III we are held for the same reason to reduce the primary i = 0.25 inches to four times, ie 1 inch, and for the reason given in § 55 above, the method with divided z is to be used. This gives, if we proceed from such a situation of the first reduction [4] , that the a occur without breakage, the following distribution tables and elements.

[4] The possibility of different reduction positions will be discussed further.

Table III reduced to different levels.
$E = 0.25$ inches; $m = 2047$; $A_1 = 71.77$.

$i = 1$ inch $i = 2$ inches $i = 4$ inches $i = 8$ inches

a	*z*		*a*	*z*		*a*	*z*		*a*	*z*
60	1		60.5	1		61.5	1		63.5	98.5
61	0		62.5	0		65.5	97.5		71.5	1815
62	0		64.5	17.5		69.5	823		79.5	133.5
63	0		66.5	80		73.5	992		87.5	0
64	2		68.5	280		77.5	129.5			
65	15.5		70.5	543		81.5	4			
66	26		72.3	626.5		85.5	0			
67	54		74.5	365.5						
68	108		76.5	113						
69	172		78.5	16.5						
70	253		80.5	3						
71	290		82.5	1						
72	330.5		84.5	0						
73	296									
74	223.5									
75	142									
76	75									
77	38									
78	13									
79	3.5									
80	2									
81	1									
82	0.5									

83	0.5									

Elements of Table III after derivation from different reduction stages .

E = 1 inch; m = 2047; A_1= 71.77.

i	1 ***E***	2 ***E***	4 ***E***	8 ***E***
A_2	71.75	71.76	71.77	71.64
C_2	71.81	71.83	71.91	71.58
D_p	71.99	72.06	-	-
D_i	72.04	71.98	72.16	71.54
u	+ 39	+ 41	+ 70	- 29
u	-120	- 147	-	-
e **,**	2.16	2.26	-	-
e '	1.92	1.96	-	-
p	0.75	0.77	-	-

As can be seen, this table confirms the conclusions drawn from the reduction stages for I.

§ 63. As for Plate IV concerning rye ears with m = 217, I have found through repeated experiments that to get to a regular kernel one can not well go down to a reduced i = 4 ***E*** , where ***E*** = 0 , 5 cm is; which, at the beginning of the panel with a reduced a = 42, gives the following results:

Panel IV, reduced to several stages.

E = 0.5 cm; m = 217; A_1 = 86.54.

i = 4 ***E*** i = 8 **E** i = 16 ***E*** i = 32 ***E***

a	*z*		***a***	*z*		***a***	*z*		***a***	*z*
42	1		44	1		48	4		56	26
46	0		52	3		64	22		88	176.5
50	1		60	8th		80	85		120	14.5
54	2		68	14		96	91.5			
58	3		76	35		112	14.5			
62	5		84	50						
66	6		92	51.5						

70	8th		100	40							
74	15		108	13							
78	20		116	1.5							
82	25										
86	25										
90	32										
94	19.5										
98	24.5										
102	15.5										
106	10										
110	3										
114	1.5										
118	0										

From this I am content to derive only the main values, which also show a very small change depending on the reduction level.

Main values of Table IV after reduction to different levels .

$\boldsymbol{E}$ = 0.5 cm; m= 217; A_1= 86.54.

i	4 $\boldsymbol{E}$	8 $\boldsymbol{E}$	16 $\boldsymbol{E}$	32 $\boldsymbol{E}$
A_2	86.48	86.67	86, 67 [5)]	86.30
C_2	87.60	87,60 [5)]	87.53	86.96
D_p	90.25	-	-	-
D_i	89.44	88.76	89.25	87.41

[5)] [The agreement of the values of A_2 for i = 8 and i = 16, as well as for C_2 for i = 4 and i = 8 is due to the nature of Table IV, namely the equality of the two A_2 follows from it that in the reduction step i = 8, the sum of the first, third, fifth z etc accidentally equal to the sum of the second, fourth z is etc, while the equinumerous *for* the stage i = 4 (for a = 82 and 86) the equality of the both C_2 condition.]

§ 64. In the meantime, apart from the choice between the reduction stages, it is still a matter of choice between the reduction situations.

The difference in the reduction positions is due to the fact that, depending on the initial value of combining the primary naked *a* , the reduced panel turns out differently. Consider this first with respect to the main inventory of the Primary Panel I. Assembling *a* began in Example 53 with the first *a* = 380 of the First Division, and we obtained as a reduced *a* 382 with the reduced *z* = 11. Let's go now consistently with it, then the reduction of the second main department with the five naked *a* 385, 386 flg. a reduced *a* = 387 with the reduced *z*= 25. But now nothing prevents the beginning of the co-venture of five bare *a* a *a* advance, thus reducing others to divisions arise, namely to stop at the first two:

naked *a*	381	382	383	384	385	386	387	388	389	390
primary z	1	2	3	3	8th	2	6	4	5	7

From which follows:

reduced

a	**intervals**	*z*
383	380.5 - 385.5	17
388	385.5 - 390.5	24

etc

This gives, as you can see, another reduced board of the main stock than the previous one, which increased primarily with *a* = 380, reduced with 382, instead of raising it primarily with 381, reduced with 383. Further, instead of raising with primary *a* = 380 or 381 , one could also raise 382, 383, or 384, and only when one starts with 385 would one fall back into the first mode of reduction, beginning with 380, with 385 beginning as a continuation includes.

On the whole, as many reduction layers are possible as the number of primary *a* or *i* , which are combined in the *i of* the reduction step. If now the i = 1mm of the primary panel I is increased to *i* = 5 mm in the first reduction stage , then five reduction layers are possible, with reduction to 10 mm ten layers would be possible, and so on. And if, in the sense of the method α) Treat primary departments with supplementation with blank *a* in unison with the main departments, so the number of reduction situations in question will be extended to them.

In order to exhaust the possible reduction positions of a given reduction step, we not only have to fill in the gaps between the primary *a* by empty *a* , but also to go back behind the first valid *a* so far and in so many ways to empty *a* that the first valid one *a* still below together increasing *a* is included, di at five possible positions according to the position, respectively with four, three, two, with a blank *a.* So in Table I, where 368 is the first valid *a* with *z* = 1 , we will have to set for the first location:

a	364	365	366	367	368
z	0	0	0	0	1

with red. a = 366 as the middle between 364 and 368, and. red. z = 1 as the sum of the red. Interval contained z ; in the second pushing forward with a a :

a	365	366	367	368	369
z	0	0	0	1	0

with red. a = 367, red. z = 1, etc., which gives the following five layers:

Panel I (vertical circumference) in five reduction positions

with i = 5 mm; $\boldsymbol{E}$ = 1 mm; m = 450.

a	***z***		***a***	***z***		***a***	***z***		***a***	***z***		***a***	***z***
366	1		367	1		368	1		369	3		370	3
371	2		372	2		373	2		374	1		375	1
376	2		377	3		378	5		379	5		380	7
381	9		382	11		383	17		384	18		385	22
386	23		387	25		388	24		389	29		390	30
391	28		392	31		393	36		394	33		395	33
396	45		397	40		398	41		399	49		400	55
401	47		402	54		403	59		404	55		405	50
406	60		407	63		408	65		409	66		410	73
411	65		412	64		413	65		414	62		415	52
416	60		417	57		418	51		419	53		420	55
421	44		422	47		423	40		424	39		425	35
426	34		427	22		428	17		429	13		430	12
431	13		432	18		433	19		434	16		435	14
436	10		437	5		438	4		439	4		440	5
441	5		442	5		443	2		444	2		445	2
446	2		447	2		448	2		449	2		450	1

To distinguish the different layers, is likely to be the easiest of designation by the beginning of the reduced panel, di the smallest reduced a or reduced E , operate, according to which that is the first of the above reducing layers by E , = 366, the second through E , = 367 usf.

[The influence of the reduction position on the values of the elements is shown in the following table:]

Elements of Table I (vertical perimeter) when reduced to five different layers.

$\boldsymbol{E}$ = 1 mm; i = 5 mm; m= 450; A_1= 408.5.

E ,	**366**	**367**	**368**	**369**	**370**	**medium**
A_2	408.6	408.7	408.2	408.5	408.6	408.5
C_2	409.1	409.1	408.6	408.9	409.1	409.0
D_p	410.7	410.5	409.7	410.4	410.3	410.3
D_i	411.0	410.1	410.5	410.2	410.1	410.4
$\boldsymbol{m}$,	246	244	240	244	242	243
$\boldsymbol{m}'$	204	206	210	206	208	207
e ,	12.3	12.1	11.9	12.1	12.1	12.1
e'	10.2	10.3	10.4	10.2	10.4	10.3
u	+13	+10	+10	+11	+16	+12
$\boldsymbol{u}$	- 42	- 38	- 30	- 38	- 34	- 36
p	0.76	0.78	0.73	0.79	0.71	0.75

Note that the A_1 of the primary table equals 408.5, and that the A_2 deviates only slightly from it in all five positions, and consequently from each other, but on the whole agrees entirely with A_1 . Likewise, the other main values show little difference, depending on the different situations; the deviations and the variance sums and the resulting mean deviations are slightly different.

But it may be remarked that as little as the values *A, C, D* differ in the same situation, they appear in the same order in all reduction positions. In fact, *D is* greater than *A,* and *C* falls between the two values, as required by the law of asymmetry. The asymmetry is also clearly evident in that everywhere $\boldsymbol{m}$, > $\boldsymbol{m}'$is; indeed, the requirement for the case of asymmetry that $p = (D - C):(D - A)$ is very approximate = ¼ π = 0.785 is fulfilled.

§ 65. While we now such a shape in Table I by virtue of increase of the primary *i* the possibility of five different panels obtained reduced to five times, we will get at III due to increase to four times the possibility of four reduction layers.

Plate III in four reduction positions

With i = 1 inch; $\boldsymbol{E}$ = 1 inch; m = 2047.

a	*z*		***a***	*z*		***a***	*z*		***a***	*z*
59.5	0.5		59.75	1		60	1		60.25	1
60.5	0.5		60.75	0		61	0		61.25	0
61.5	0		61.75	0		62	0		62.25	0
62.5	0		62.75	0		63	0		63.25	0
63.5	1		63.75	2		64	2		64,25	4
64.5	8th		64,75	11.5		65	15.5		65,25	18.5
65.5	20		65.75	22.5		66	26		66.25	35
66.5	41.5		66.75	43.5		67	54		67.25	60
67.5	72		67.75	94		68	108		68.25	123.5
68.5	137		68.75	151.5		69	172		69.25	192
69.5	215.5		69.75	237.5		70	253		70.25	263.5
70.5	271		70.75	280		71	290		71.25	309
71.5	323.5		71.75	327		72	330.5		72,25	318
72.5	305		72.75	304		73	296		73.25	285.5
73.5	274.5		73,75	248.5		74	223.5		74.25	205.5
74.5	183.5		74.75	165		75	142		75.25	119
75.5	101.5		75.75	87.5		76	75		76.25	62
76.5	52		76.75	43		77	38		77.25	35
77.5	27.5		77.75	18.5		78	13		78.25	9.5
78.5	7		78.75	5		79	3.5		79.25	3
79.5	3		79.75	3		80	2		80.25	1.5
80.5	1.5		80.75	1		81	1		81.25	0.5
81.5	0		81.75	0		82	0.5		82.25	1
82.5	1		82.75	1		83	0.5		83.25	0

Elements of Table III after reduction in four layers.

$\boldsymbol{E}$ = 1 inch; i = 1; m= 2047; A_1= 71.77.

E ,	**59.5**	**59.75**	**60**	**60.25**	**medium**
A_2	71.76	71.75	71.75	71.76	71.755

C_2	71.79	71,80	71.81	71,80	71,80
D_p	71.91	71.96	71.99	71.97	71.96
D_i	71.74	71.92	72.04	71.97	71.92
u	+ 21	+33	+ 39	+ 28	+30
$\boldsymbol{u}$	- 76	-104	-120	-106	-101.5
H	2.05	-	2.04	-	2,045
$e_,$	2.12	2.14	2, 16	2.15	2.14
e'	1.97	1.93	1.92	1.94	1.94
p	0.80	0.76	0.75	0.81	0.78

It can be seen that the results of the previous Table I are best confirmed by those of Table III. Here, too, D_i almost always agrees with D_p, with the exception of the position $E_,$ = 59.5, where, quite exceptionally, D_i deviates not only relatively strongly from D_p, but also against the direction of the essential asymmetry smaller than A_2 and C_2 is.

§ 66. [Since for Table IV the reduced $i = 4\,\boldsymbol{E}$, which is due to its irregularities, *is* the primary i but = 0.1 $\boldsymbol{E}$, in this case basically 40 reduction positions are possible. From these, the following four layers should be selected:

Plate IV in four reduction positions

with $i = 4\,\boldsymbol{E}$; $\boldsymbol{E}$ = 0.5 cm; m = 217.

a	*z*		***a***	*z*		***a***	*z*		***a***	*z*
41	1		42	1		43	1		44	1
45	0		46	0		47	0		48	1
49	1		50	1		51	2		52	1
53	1		54	2		55	1		56	2
57	3.5		58	3		59	3		60	4
61	5		62	5		63	7		64	6
65	3.5		66	6		67	7		68	8th
69	9		70	8th		71	9		72	9
73	11		74	15		75	17.5		76	21.5
77	23.5		78	20		79	18.5		80	15.5
81	19		82	25		83	21		84	24
85	23		86	25		87	30		88	33.5

89	35.5		90	32		9 [1]	30		92	27.5
93	22		94	19.5		95	22.5		96	23.5
97	24		98	24.5		99	22		100	18.5
101	18		102	15.5		103	13.5		104	13.5
105	12		106	10		107	8th		108	4
109	2		110	3		111	4		112	3.5
113	3		114	1.5		115	0		116	0

Elements of Table IV after reduction in four layers.

E = 0.5 cm; i = 4; m= 217; A_1= 86.54.

E ,	41	42	43	44	medium
A_2	86,50	86.48	86.59	86.52	86.52
C_2	87.90	87.60	87.87	87.85	87.805
D_p	90.19	90.25	91.31	90.58	90.58
D_i	88.92	89.44	89,00	88.45	88.95
u	- 41	- 41	-52	- 45	- 45
e ,	11,70	11.86	12.28	11.82	11,915
e'	8.01	8.09	7.56	7.76	7,855
p	0.62	0.70	0.73	0.67	0.68

This table also shows the relative constancy of the principal values and deviation functions in the different positions, the regularity in the succession of A , C and D , as well as the proximity of D_i and D_p when the main values are more divergent than in I and III . However, p *is* consistently smaller than the theoretically required value of 0.785.]

§ 67. Now the question arises as to which of the various reduction situations one has to adhere to in deriving the elements or examining the established laws, about which again quite general, fixed rules can not be given, but the following general statements will be made ,

First, our panels can be shown without saying that with so large by the appearance *m,* subject as our panels, the changes of the elements depending on the reduction position are irrelevant and generally of the order of the uncertainty that at all permissible in determining the elements is, so that it does not seem to make much difference to which reduction position one will hold, and only has to observe the rule of determining all elements which are to be examined in the same reduction position. Occasionally, however, it happens that under different reduction situations, one or the other a disadvantage against the rest in regard to the regular gear of *z* shows how z. For example, among our five panels I (§ 64) the last one gives $E_,$ = 370 a deviation from the regularity, provided that it receives the consequence of the reduced z: 55,50,73, instead of the *z*should rise to a maximum of 73 continuously. All the other four panels, on the other hand, show nothing of the kind and are therefore preferable to them. This suggests that, if you happen to hit a core with an irregular path, you can see if you are not going better with another layer. In general, when comparing different reduction positions, the one which shows the least deviation from the general laws of distribution will have to be chosen. Incidentally, any choice could be avoided by taking into account the possible reduction situations and drawing the remedy out of the results, except that this would be troublesome to carry out and would lead to less rewarding inconveniences.

If we now take a comparative look at the value of primary and derived reduced tables, it follows that for complete treatment of a K.-G. rather, they must complement each other, rather than replace them, according to which it is only to be regretted that the great space generally occupied by the primary panels is usually compelled to renounce their communication and to be satisfied with reduced ones. In any case, in the primary table one has the direct empirical basis for the whole treatment of a given K.-G., and since the reduction is based on the size of the *i,* the position of the intervals, after whole and halved *z* Either way, anyone who chooses the primary board will still be free to choose which option to take, and he will have the option of modifying and controlling a choice that has already been made. Also, the arithmetic mean can not be obtained as surely by a reduced panel as from the primary, although the difference in many items may be negligible. Against this one can follow the legal course of the values of a K.-G. a general reduction of the table and in determining the elements that are involved in local irregularities in a special way, not at all a local reduction, and the reduction of the panel will in any case have the advantage of proving a regularity,

IX. Determining $\sum a$, $\sum a_,$, $\sum a'$, $m_,$, m', $\alpha\theta_,$, $\alpha\Theta'$.

§ 68. To explain the application of the rules to be given below, each of the previous distribution tables could serve. It simplifies and hereby simplifies the application of the rules with the brevity of the panels, and so I will first follow a small panel constructed only from the general scheme of a collective distribution panel , by the way arbitrarily, from only eight *a ofthe a-* column The following explanations may tie together, which, properly put, then apply to any real collective distribution panel application. The columns $S_,$, S' are auxiliary columns, which are obtained immediately their explanation.

Small, arbitrarily arranged distribution panel .

$i = 2$; $m = 80$; $\sum a = 912$.

a	**intervals**	*z*	*za*	*S,*	*S '*
3	2 - 4	1	3	1	80
5	4 - 6	2	10	3	79
7	6 - 8	5	35	8th	77
9	8 - 10	10	90	18	72
11	10 - 12	30	330	48	62
13	12 - 14	20	260	68	32
15	14 - 16	10	150	78	12
17	16 - 18	2	34	80	2
total		80	912	304	416

In the previous table the meaning of the values in the columns *a,* Interv., *Z, za is known* according to the earlier explanations, but the values *S* , ' *S'* are explained as follows: The first *S* , is equal to the first *z* , the second *S* , equal to the first + second *example,* the third equal to the first + second + third *z* , etc., so that the last is equal to the sum of all *z* and hereby = *m* is. After this, each, given *a, is a* corresponding *S* , by summing the *S* belonging to the preceding *a* , with the *z of* the relevant *a* .

The same procedure is used in the column of *S* ′, but with the summation from below in the opposite direction.

§ 69. Now, apart from the total sum $\sum a$ and the total number *m* , a crude and a sharp determination of the values concerned must be distinguished in the sense already given; a raw, if one calculates, as if the number *z,* written on each *a of* a primary or reduced plate, belongs entirely to it; a sharp, provided that it is taken into account that they are actually within the perimeter interval of each *a*it is to be thought of as distributed, according to which the value of the elements to be determined in the interval in which the determination of the same intervenes shortly after the intervention interval, is to be determined interpolation, as shown below. So far, one has not entered into this; In the following it will be to be discussed and the advantage of it proven.

The to be interpolated at sharp determination interval, so-called. Intervention interval, I will of its location and size of generally *I* call. In our example table, it is, according to the size of the table through the table *i* = 2, that its position can change according to the nature of the task. In general, let his first boundary be g_1, which results from the column of intervals, and his second, g_2; that is, when 10-12 is the engagement interval, $g_1 = 10$, $g_2 = 12$.

In addition, be general:

z_o the value *z,* which is based on the engagement interval *I* falls,
a_o in the column of *a* the relevant *I* associated value of *a,* which the center of *I* is,
$z_o.a_o$ the demgemäße Dimension product which on *I* falls,
v the so-called prefix, ie the sum of the *z* and ***V*** the sum of
the *za,* which extends from the beginning of the table in to the beginning of *I* ,
n the so-called suffix and***N*** sums, which reaches from the conclusion of*I*to the end of the table,
$x = H - g_1$, measure of intervention, the value by which theprincipal
value*H* fallingin*I*gives the beginning of*I,*dig_1,
$y = m, - v$, engaged number, the number by which the up from the beginning
beginning*H* number ranging up to the beginning of*I*-reaching presented,
*Y*engagement sum, the sum of*a*, which from the beginning of the*I*up to*H*ranges.

Generally you have:

$$v + n + z_o = m,$$

$$V + N + z_o a_o = \Sigma a = \Sigma za.$$

Now, if for the following explanation the interval 10 - 12 will introduce our *I* , we have:

$$m = 80; \Sigma\, a = \Sigma\, za = 912;$$

$$g_1 = 10; g_2 = 12;$$

$$z_o = 30; a_{o =} 11 ; z_o a_o = 330;$$

$$v = 18; V = 138;$$

$$n = 32; N = 444;$$

$$x = H - 10; y = m \;,\;\; - 18.$$

Any value may occur as *H* , but we will tie the explanation preferably to the arithmetic mean of the table as *H* , which is found by dividing $\Sigma\, za = 912$ by $\Sigma\, z = 80$ equal to 11.4, and hence $x = 1, 4$ gives; but the central value should also serve as *H*

§ 70. Determination of a sum of value Σ *a.*

This determination is made directly by summing the *za,* so that Σ *a is* used synonymously with Σ *za* .

With tables as small as our example board, the formation and summation of *za* makes no difficulty; but if a panel expires far that *a* of *a* column and hereby forming Maßprodukte *za*are very numerous, especially meets the primary panels, this formation and summation is extremely complicated and easily subject to calculation error. Try only one of our primary tables; and even in the reduced tables the same inconvenience, albeit to a lesser degree, still asserts itself. Therefore is highly desirable that an on primary reduced as tablets each stage and location just applicable method disposal is, $\sum a$ (and thereafter *A)* Finding all the same value, but in a far more convenient manner than according to the previous method which I that of *such a* will call, while the I folgends auseinanderzusetzende which the *S* call. It belongs only to what is not essential for the procedure of *za* , that the tables to which the method of *S is to be* applied are made equidistant or equidistant by the use of empty *a* , according to which one can confine themselves to the inconvenient method of *za* be limited to cases where the equidistance is not established.

One can now use any of the $S_,$ or S' to determine the sum $\sum a$. At first, the determination is made according to the following formula:

$$\sum a = Me' - Z_, \quad i \,; (1)$$

otherwise, according to the formula:

$$\sum a = mE_, \quad + \quad Z'i. \ (2)$$

Therein the letters have the following meaning. Under *m* the total number of *a is* understood, whose sum is to be taken, ιε $\sum z$, under E' the largest *a* or upper extreme (which is indeed in the table below), under $E_,$ the smallest *a* or lower extreme among them *a* , which values when about to be summed *a* should merely a piece of a whole distribution panel, are found to relate only to the piece, not the whole panel. Further, let $Z_,$ the total sum of $S_,$ Which the to be summed *a* belonging, less the *south* $_,$ that to *'E* belongs, or what says the same, the total sum of $S_,$ excluding the extreme *south* $_,$; Further, Z' the sum total of S' exclusive of what to $E_,$ belongs; *i is* the constant difference around which the *a of* the *a*column diverge.

Now let us take the $\sum a$ *of* the whole example-table, then $m = \sum z$ *of the* same 80; $E' = 17$; $E_, = 3$; $Z_, = 304 - 80 = 224$; $Z' = 416-80 = 336$; $i = 2$. If one uses the first or second formula, one will find, according to these values, $\sum a = 912$, in agreement with the directly determined sum of *za* , which stands under the column of *za* .

Quite the same way the sum can $\sum a$ for each piece of the sample panel found, only that the values of *m* , E', $E_,$, $S_,$, S' have to amend accordingly, just as if the summation only for the four *a* of *a* column would have to action from 5 to 11, one would have: $m = \sum z = 47$, $e' = 11$, $e_, = 5$, $i = 2$. the columns of $S_,$, S' , but would be to form:

	$S_,$	S'
	2	47
	7	45
	17	
40 47 30	Total: 73	162

hence $Z_, = 73 - 47 = 26$; $Z' = 162-47 = 115$;

what gives: $\sum a = 465$.

For very long series one may find it inconvenient to have to progress to very large values of S in the progress ; which, however, can easily be remedied by dividing the whole series into two or more departments, and whose $\sum a$ seeks in the former way, and finally to unite them. But even more practical is the combined application of the column $S_,$ and S' in the following way.

In particular, somewhere, most conveniently about the middle of the tablet, write a value a, which is called c, add the column of $S_,$ up to this c, excluding it, and also the column of S' excl. C, sum up the S thus obtained $,$ as S' especially; the first sum is called $Z_,$ the second Z', as it used to be, then one has:

$$\sum a = mc + (Z' - Z_,)\, i, \quad (3)$$

resulting in:

, (4)

where m *is* the total number of all to be summed a.

§ 71. I have found the S method in an American treatise on the measures of the recruits (by ELLIOTT) 1) without specifying how the author came to do so, and without proof of its generality. Now this proof can certainly be given, but, although elementary 2), it is rather cumbersome and difficult to follow; I pass it therefore, since the method consists of every empirical test, but adds the following remarks to assure its application.

1) [EB ELLIOTT, On the military statistics of the United States of America. Berlin 1863. (International statistical congress at Berlin). S. note on the construction of certain tables, p. 40.]

2) [Indeed. merely necessary, $\sum za$ *in* more detail by $z_1 a_1 + z_2 a_2 + z_3 a_3 +.,z_n a_n$ and represent the equidistants $a_2, a_3, \ldots a_n$ by $a_1 + i, a_1 + 2i, \ldots a_1 + n - 1 i$ to replace, by properly contracting the terms, the transformed sum in the form: $a_1 (z_1 + z_2 + \ldots z_n) + i(z_2 + z_3 + .z_n) + i(z_3 + \ldots z_n) + \ldots iz_n$ and thus the empirical formula: $E_, m + Z'i$. Similarly, $E'm - Z_, i$, if one $a_1, a_2, a_3 \ldots a_n - 1$ resp. is replaced by $a_n - n - 1 i, a_n - n - 2 i, a_n - n - 3 i \ldots a_n - i$.]

1. Of course, the correctness of the determination of $\sum a$ and sequentially of A depends on the correctness of the S columns. If an S *is* wrong in the order, all subsequent ones will also be wrong, because each earlier S *will be included* in all later ones, and when ascending to high values of S, an error may easily occur. But

one has a light and unavoidable means of control in that, when an *S* column is used, the extreme *S* , which does not occur in *Z* , must agree with *m* ; in the combined method of *S* , and *S* ′ but the last, in *Z* not with incoming values of *S* , and *S* ′, to which one arrives, with the *z* value of *c* must give the total number *m* .

2. The *S* -method is equally applicable to panels with and without empty void *a* , and the formation of the *S* column is done according to the same rule; but it will certainly be useful to explain especially the application of the rule for the case of empty *a* with *z* = 0 in order to anticipate possible misunderstandings and consequent oversights. According to the given rule, every *S* belonging to a given *a* *of* the *a* column becomes the sum of the *S* belonging to the preceding *a* with the *z* *of* the relevant one*a* received. If the latter *a is* an empty with *z* = 0, then, of course, according to the previous rule, its *S is* a mere repetition of the preceding S, and so many empty *a* follow each other, so often does the *S of* the preceding valid *a* repeat itself . Our two examples (in § 68 and § 70) give no explanation for this, since, like most reduced tables, they contain no empty *a* ; the more opportunities are offered by the primary panels, especially in their end departments. For a brief explanation, however, we also put here a small board with some empty *a*and arbitrarily cling to the repetitive *S* associated with the empty *a* for easier distinction from the others, but that they can not be excluded from the summation in the formation of ∑ *S* and thus *Z* , since they are much more like the others:

a	*z*	***S,***	***S*** ′
3	2	2	50
5	0	(2)	(48)
7	0	(2)	(48)
9	10	12	48
11	30	42	38
13	5	47	8th
15	0	(47)	(3)
17	3	50	3
total	50	204	246

If, as often in the Endabteilungen primary panels, a larger number of empty, *a* , and therefore repeated bracketed *S* follow one after the other, you will find it easiest to these bracketing equal in sum, except that one has to guard against the subsequent *S* then not to be determined as the sum of this sum of *S* with the new *z* , but as the sum of the naked *S* preceding the engagement with the new *z* . Thus, the series of $S_,$ previous panel, will take the form:

2, (4), 12, 42, etc,

that is, the $S_,$ = 12 belonging to *a* = 9 with *z* = 10, can not be formed by adding 10 to the preceding summed (4), but to the naked 2 preceding the switch-on; a rule that should be considered. Applying this now to the entrance of our primary tablet I (chapter VII), the required (feasible) execution of the empty *a, of* which two between 368 and 371, four between 371 and 376, one between 376 and 378 fall, the series of $S_,$ so make:

1, (2), 3; (12); 4, (4), 5, 6, etc

In the primary panel III, where *i* = 0.25 inches, between the first two valid *a,* di 60 and 64 whole inches respectively, with *z* = 1 and 2, even 15 empty *a* , further between 64 and 64.75 two , and the beginning of the $S_,$ series looks like this:

1 , (15), 3, (6), 7, etc.

It is important to be with. to be well acquainted with this use of the empty *a* , and to control its proper execution in every case by careful revision, because one is too easy in it, and because the above control of the proper formation of the *S* -columns, that their last Value with *m* coincide, even with the use of empty *a* still must apply, so is not negligible, but also, if it does not ensure against an incorrect use of the empty *a* .

§ 72. Determination of the lower and upper sums, resp. $\sum a_,$ and $\sum a'$, with respect to a given principal value *H.*

For example, let *A* main value, in our example, Table 11.4, then one has to all raw determining *a,* which are smaller than 11.4, that is from *a* = 3 to incl. *A* = 11 to be summed, ie, the corresponding *za* to sum to $\sum a_,$ to have; however, one obtains $\sum a'$ by summing the *za* from *a* = 13 to the end, ιε $\sum a_, = 468$, $\sum a' = 444$. Except by directly summing the *za*These sums can be obtained in the manner indicated by the *S* method.

For sharp determination one has to think of the sum $\sum a_,$ composed of two parts, the sub-sum V , which extends from the beginning of the table in to the beginning of the intervention interval *I* , and the mesh sum *Y* , which from there up to *H* , our case until *A* , ranges and is obtained by simple interpolation, by setting that the engagement sum *Y* to the total sum of the interval *I* , di to $z_0 a_0$, behaves as the intervention measure *x* to the total interval *I* ; therefore:

$$Y : z_0 a_0 = x : I, \quad (5)$$

so:

; (6)

hereafter:

, (7)

In our example table, $\boldsymbol{V}$ = 138, $z_0 a_0$ = 330 , x = 1.4, I = 2; therefore:

$\sum a_{,}$ = 369; $\sum a'$ = $\sum a$ - $\sum a_{,}$ = 912 - 369 = 543,

which is very different from the crude regulations.

Should any other principal value H occur instead of A , the previous formulas would remain the same, except that x *should be taken* instead of = A-g_1 , rather, H = g_{-1} . Be e.g. the sharp certain C as H taken. According to § 82, for our table, with rounding off in the last decimal, there is little, but slightly different from A , equal to 11,467, hence x = 1,467; gives:

$\sum a'$ = 912 - $\sum a_{,}$ = 532 - 0.055,

where the small additions to 380 and 532 need to be omitted, because they depend only on rounding off the C in the last decimal.

[One wanted now; In order to obtain an even more accurate determination of the intervention *sum Y* , instead of the simple interpolation, a sharper, by pulling in second differences, let occur, this would not be allowed. For the products *az* , which are to be used as first differences, represent the sum of the values *a* falling on an interval *i* only on the assumption that these values are distributed uniformly over the whole interval. It is thus by this way of thinking the dependence of the engagement *sum Y* of the engagement measures *x* already regulated and in particular influenced by the product values *az* preceding or following the interpolated intervals *,* as would have been assumed in the case of the addition of second or even higher differences. If, therefore, from the same point of view, the summation of all falling on a whole interval *a* subject; determine the meshing *sum Y* with the greatest possible sharpness, one must take the values *a* involved in the formation of the meshing sum , whose number is the meshing number *y* and in the following paragraph is equal to $z_0 x : I$, in the middle of the meshing dimension *x*designated subinterval, which think in $g_1 + ½ x$, united and thus

(8th)

place as described above, equal to $z_0 a_0 x: I$ bet. The sum of the *a,* one finds then the same:

, (9)

where the index attached to the sum sign may serve to distinguish it from formula (7). In proportional determination of *Y* therefore is $\sum a_,$ by the amount

Too much has been taken into account, so that the more precise way of determining (8) will generally give an advantage which may be considered. In fact, for the A = 11.4 of our example ταβλε , we get $\sum' a' = 362.7$ versus $\sum a_, = 369$.]

[But if the accuracy so achievable is not satisfied, then not only *Y* but also ***V*** and ***N are*** to be determined on the basis of the idea that instead of the uniform distribution of the *a*within the individual intervals, a continuous contraction, taking into account the neighboring intervals changing occurs. Thus one achieves the next higher degree of accuracy, if the addition of the neighboring intervals to one of the two directly adjacent intervals, e.g. B. limited to the, as it progresses from the smaller to the larger *a* immediately following interval. Then the previous provisions are to be replaced by the following.]

[Designates z_1, the number of values falling in the following the procedure intervals interval, and are added to the values of the first, the Extreme $E_,$ belonging, and the values of the last, the extreme of E' not pull enclosing interval into account having to, at the beginning and end of the panel an empty interval with $z = 0$ added, then the sum of the determined *a*the whole engagement interval equal to $a_0 z_0 - {}^1/_{12} I (z_0 - z_1)$, which Vorsumme equal to $\boldsymbol{V} + {}^1/_{12} I z_0$, the Nachsumme equal to $\boldsymbol{N} - {}^1/_{12} I z_1$, where ***V*** and ***N*** are calculated as above, and the total sum of *a* is thus equal to the calculated above $\sum a$. To calculate the intervention sum, the formula also serves:

, (10)

from which finally:

(11)

follows.

§ 73. Determination of the deviation numbers $m_,$, m'.

After a rough determination we find $m_,$ easily by adding together the values z belonging to the values a smaller than H; and taking $A = 11.4$ for H in our example table, this gives $\mu_, = 48$ and $\mu' = m - \mu_, = 80 - 48 = 32$.

It is the sharp determination that shall $m_,$ composed of the Vorzahl $\boldsymbol{v}$, which up to the beginning of I ranges and the engagement number y, which thence to H ranges. But this is obtained by knowledge of $x = H - g_1$ by interpolation on the basis of the proportion:

$$Y : z_0 = x : I, \quad (12)$$

therefore:

(13)

and hereafter:

, (14)

Suppose for H the value of $A = 11.4$ and thereafter the above values $\boldsymbol{v} = 18$; $x = 1.4$; $z_0 = 30$; $I = 2$, we obtain $\mu_, = 39$, $\mu' = 80 - 39 = 41$, a determination which, in turn, is very different from the raw one, and indeed causes the preponderance to fall on the opposite side.

If m' should not be directly determined by subtracting the $m_,$ from m, but what can be useful for the control, then one has in general:

, (15)

which with reduction of $H = A$ virtue of $\boldsymbol{n} = 32$ to

returns.

Be held A rather C than H taken. After sharp determination in X. Chap. For our example table we find little, but slightly different from A, equal to 11,467, hence $x = 1{,}467$, whereas the remaining values remain the same as for A. This gives:

,

Both values are, as it corresponds to the concept of the central value, equal to each other, equal to ½ $m = 40$, in that the small positive and negative addition to it again only depends on the rounding of the C in decimals.

[This determination of the intervention number y by simple interpolation has to be considered exact, as long as the distribution of the a within the individual intervals may be assumed to be uniform. However, if this is not the case, then by sharp interpolation, using second and higher differences, any degree of accuracy can be achieved. For the intermittent summarization of the numbers of the a to the z values, which are to be based on the interpolation as the first differences, is not like the corresponding combination of the sums of a to the za values of a certain assumption on the distribution of $a$within the associated intervals. Thus, when second differences are added, ie taking into account the interval immediately following the intervention interval, the *z of* which is set equal to z_1 as above , the formula:

, (16)

But taking into account, moreover, the immediately preceding interval whose *z is expressed* by z_{-1}, the formula used for the calculation of *y is* :

(17)

in which third differences are involved. It should be noted that such an intensification in the calculation of y , the corresponding tightening in the calculation of *Y,* ***V*** and ***N*** conditionally. In particular, the use of formula (16) results in the entry into force of formulas (10) and (11).]

§ 74. Determination of the mutual deviation συμσ $\sum\Theta'$, $\sum\Theta_{,}$ bez. of a given main value *H.*

Directly we get the positive variance $\sum\Theta$ 'bez. an arbitrary output value *H,* if we sum the individually determined differences $\Theta' = a' - H$; to participating folgends always absolute values of negative deviation sum $A\Theta_{,}$ if we individually determined differences $\Theta_{,} = H - a_{,}$ sum; but the individual determination of the many differences is laborious and is easily subject to individual oversight; Both are met by determination according to the following formula:

$$\sum\Theta' = \sum a' - m'H$$

$$\sum\Theta_{,} = m_{,}H - \sum a_{,} \quad (18)$$

In fact, the sum of the positive $\Theta_{,}$ di $\sum\Theta$ ', is obtained by τακινγ the value H of each of the m' values a' , ie of a , which are greater than H , that is, throughout m' times H of $\sum a$ *'is*deducted [3] ; what the first equations above are. On the other hand, the sum of the negative is Θ obtained by absolute values when the sum of $m_{,}$ values of $a_{,}$, d . i. the values *a*Which is smaller than H , are of $m_{,}$ n times H is withdrawn, which is the second of the above equations.

[3] Not of *m'a* , which could only happen if all *a had the* same size.

These formulas apply both to raw and to sharp determinations, but with the difference that for raw determination $m_,$ and m', $\sum a_,$ and $\sum a'$ raw, are determined sharply for sharp determination. Now taking A again as the principal value for our example table, in which case μ substitutes for m , Δ for Θ , we can use for raw and sharp determination of the previously determined values, according to which raw $\mu_, = 48$; $\mu' = 32$; $\sum A_, = 468$; $\sum a' = 444$; gives:

raw

$$\Sigma\Delta_, = 48 \cdot 11.4 - 468 = 79.2$$

$$\Delta' = 444 - 32 \cdot 11.4 = 79.2$$

Both sums are equal according to the concept of the arithmetic mean. After sharp determination one has $\mu_, = 39$; $\mu' = 41$; $\sum a_, = 369$; $\sum a' = 543$; hereafter:

sharp

$$\Sigma\Delta_, = 39 \cdot 11.4 - 369 = 75.6$$

$$\Sigma\Delta' = 543 - 41 \cdot 11.4 = 75.6$$

So again equality of both sums, except that the sharply determined sums are smaller than the raw determined. However, if instead of the proportional computation of Y one uses the more accurate one given above, one συβστιτυτεσ $\sum' a_, = 362.7$; $\sum' a' = 549.3$, we obtain, if here too an index is added to the sum sign to distinguish it from the above deviation sums:

sharp

$$\Sigma'\Delta_, = 39 \cdot 11.4 - 362.7 = 81.9$$

$$\Sigma'\Delta' = 549.3 - 41 \cdot 11.4 = 81.9,$$

hence two equal sums greater than the raw ones.]

This result is bez. A as H generally taken, namely:

1.in the case that $A > a_0$, thus :

sharp $\Sigma\Delta_,$ = raw $\Sigma\Delta_,$ - = raw $\Sigma\Delta_, - k$ (19)

[sharp $\Sigma'\Delta_,$ = raw $\Sigma\Delta_,$ + = raw $\Sigma\Delta_, + \kappa$]

2.in the case that $A < a_0$, with ::

sharp $\Sigma\Delta$, = raw $\Sigma\Delta$, - = raw $\Sigma\Delta$, - l (20)

[sharp $\Sigma'\Delta$, = raw $\Sigma\Delta$, + = raw $\Sigma\Delta$, + λ]

The somewhat cumbersome and penibeln proof 4) hereof I go on, but you can the accuracy of the formula at any homemade examples such. For example, confirm with our sample table. Here is $A = 11.4$; $a_0 = 11$, and consequently $A > a_0$, at the same time is $I = 2$, $x = 1.4$, therefore, $x > \frac{1}{2} I$. So the first case is. Now we had raw $\Sigma\Delta$, = 79.2. The value k to be deducted from this in order to arrive at $\Sigma\Delta$, but is calculated according to the above expression with consideration that $z_0 = 30$, to ½ ξ 30 ξ 0.6 ξ 0.4 = 3.6 and subtracting this from 79.2 gives 75.6 as found above according to the formula. [Further, the value κ , leading to $\Sigma'\Delta$, is found to be equal to $\frac{1}{4} \cdot 30 \cdot 0.6^2 = 2.7$, and this adds up to 79.2 gives 81.9, as it should.]

4) [It follows, together with the extension for any principal value H, with respect to which the lower and upper deviation συμσ $\Sigma\Theta$, resp. $\Sigma\Theta'$, by direct calculation from the formulas:

raw $\Sigma\Theta$, $= (\boldsymbol{v} + z_0) H - (\boldsymbol{V} + a_0 z_0)$,

if $H > a_0 = \boldsymbol{v} H - \boldsymbol{V}$, if $H < a_0$;

sharp

focus ,

with analogous formulas for the upper variance sums.]

Only in the special case, the difference between $\Sigma\Delta$, raw and $\Sigma\Delta$, sharp disappears , where A coincides with one of the two boundaries of I or with its center, where $x = 0$ or $= I$ or $= \frac{1}{2} I$; whereas after a maximum equation the difference in the maximum, when the first case $x = \frac{3}{4} I$, second if $= \frac{1}{4} I$ by both if the value of $^1/_{16} \cdot z_0 I$ obtained. [The difference between $\Sigma\Delta$, crude and $\Sigma'\Delta$, sharp, if A with one of the two limits of the I coincident, whereas this difference its maximum value $^1/_8 z_0 I$ are obtained when A in the middle of Ifalls.] So floats the whole difference k or l 0 interlocutory and $^1/_{16} z_0 I$ [the difference κ or λ interlocutory 0 and $^1/_8 z_0 I$]; in general, however, the difference stands for the same I and$x$ in a simple ratio to z_0.

It can now be seen that the sharp $\Sigma\Delta$, [resp. $\Sigma'\Delta$,] can also be determined by first determining the raw one that is easier to find, then subtracting *k* or *l* from it [resp. κ or λ added to it], depending on $A > a_0$ or $A < a_0$.

If *H* does not equal *A,* then one has to expect inequality instead of equality of both sums. Take z. B. *C.* The forms for their determination are here:

(21)

,

After Ch. X. *C will result* for our example table after sharp determination = 11.467, while ½ *m* = 40. And if we now also determine Σa, and $\Sigma a'$ according to the given rule, we obtain:

$$\Sigma\Theta, = 40 \cdot 11{,}467 - 380 = 78.7$$
$$\Sigma\Theta' = 532 - 40 \cdot 11{,}467 = 73{,}3$$
$$[\text{resp. } \Sigma'\Theta, = 40 \cdot 11{,}467 - 374{,}13 = 84{,}5$$
$$\Sigma'\Theta' = 537.87 - 40 \cdot 11.467 = 79.2.]$$

§ 75. Let us now apply the application of previous destinations to one of our K.-G. and we examine how far the sharp determination advantages over the raw granted in regard to the coincidence of elements in derivation from different reducing layers, so that they in respect of the determination of shows μ, (from which $\mu' = m - \mu$, followed by) is very significant, as to $\Sigma\Delta$, (which εθυαλσ $\Sigma\Delta'$) but lacks or remains doubtful, [in respect of $\Sigma'\Delta$, on the contrary, stands out noteworthy].

I made the rather arduous comparison at the 5 reduction positions of the distribution panel of the skull vertical, which are explained in § 64, and whose sharply defined elements are listed there.

Comparison between the raw and sharp certain values of μ, and AΔ, .

E,	366	367	368	369	370	medium	Σdiff.
A	408.6	408.7	408.2	408.5	408.6	408.5	0.7
μ, raw	217	230	250	193	201	218.2	87.2
μ, sharp	218	220	220	219	217	218.8	5.2
$\Sigma\Delta$, raw	2531	2509	2471	2492	2531	2,506.8	101.2
$\Sigma\Delta$, spicy	2528	4292	2465	2479	2509	2,494.6	95.6
$\Sigma'\Delta$, sharp	2531	2513	2505	2518	2540	2,521.4	56.4

The column $\sum$ diff. gives the sum of the deviations of the 5 individual determinations from the mean determination, and herewith a kind of scale for the variation depending on the situation. The disadvantage raw against sharp for μ , is hereafter in fact tremendously, for AΔ , too low to no doubt to stay [for $\sum '\Delta$, however, sufficiently large to make the following the precise determination manner appear to be favorable]. By the way, you can notice that the situation E ,= 370 perhaps better excluded from the comparison, because the distribution table of this position according to § 67 shows an anomalous irregularity in the nucleus, which makes it not well applicable for the calculation of the elements.

The primary table is not included for comparison, because it does not permit any definite determination in the large irregularity and nonuniformity of the estimate. However, one might ask, if not the *A of the* same = 408,5 for the derivation of all μ , and $\sum\Delta$, in the 5 layers is preferable, because the reduction brings no advantage, but rather a slightly greater uncertainty in the determination of the *A*. However, I do not consider this to be appropriate for the following reasons.

In any case, for the derivation of the other principal values as *A* , the disadvantage of the irregularity and equality of estimation of the primary chalkboard is predominant, and one must nevertheless adhere to a reduced chalkboard, and then, in my opinion, also deduce *A* from the same reduction stage and position, which is assumed to be reducible is not to alter the ratios of the various principal values by the inconsistency in this regard. In any case, there is usually only a reduced table for the derivation of the *A* and the other elements. Incidentally, the *A of* the reduced plates according to the results of compilations § 64-66 of the primary *A*generally differs little; Nor can any significant difference be expected from following one and the other procedure. In this respect I have examined at least μ , comparatively on the same table which gave the previous results applying the 5 special *A* for derivative of μ , by deducing the same everywhere from the primary *A* = 408.5, and obtained the following results According to which μ , raw has not changed anywhere before, it has sharply changed μ , so that the agreement between the different layers is somewhat reduced, provided that $\sum$ diff. previously was only 5.2, 11.6 folgends is what common ground only to the detriment of the performed application of the primary *A* with respect to the particular application of the reduced *A* can be interpreted.

E ,	366	367	368	369	370	medium	$\sum$ diff.
μ , raw	217	230	250	193	201	218.2	87.2
μ , sharp	217	217	224	219	216	218.6	11.6

Starting with the mean deviation, by doubling $\sum\Delta$, first $\sum\Delta$ and hereafter:

and . (22)

Untriftig it would be, as is done ELLIOTT in his treatise on American recruits extent η as a means of $\eta_{,} = A\Delta_{,} : \mu_{,}$ and $\eta' = A\Delta' : \mu'$ ½ (di = $\eta_{,}$ + η to want to determine '); for not only does this run counter to the meaning of the original Gaussian rule, but one also neglects the various weights which come to $\eta_{,}$ or η, depending on their derivation from $\mu_{,}$ and μ 'values; what the right mean:

(23)

is.

X. Compilation and correlation of the main properties of the three main values *A, C, D,* further *R, T, F.*

§ 76. In addition to throughout my preferred three main values, the arithmetic mean of *A,* the median *C* and the closest value *D* are folgends three are beside the point considered by me, which I as a separator value *R,* heaviest value *T* and deviation focal value *F* performers.

Clearly arranged according to their main differences, they are the following.

Severity *R* , the value *a* , in relation to which $\sum a' = \sum a_{,} = ½ \sum a$, hence the sum of the larger values is equal to the sum of the smaller ones and therefore each of them is equal to half the total sum of *a* .

Arithmetic means *A* , the value of *a* with respect to which $A\Theta' = A\Theta_{,}$ ie the sum of the positive deviations is equal to the sum of the negative; and bez. whose $\sum\Theta^{2}$ is a minimum.

Central value *C* , the value *a* , with respect to which $m' = m_{,}$ ie the number of positive deviations equal to the number of negative deviations, and $\sum\Theta$ ισ a minimum.

Densest value *, D* , the value *a* with respect to which the deviation numbers of both sides $m_{,} : m'$ as the mean errors of the same $e_{,}$ e' behavior, and the measured value *z* is a maximum.

The heaviest *a* value T, the value *a* , whose measured product *za is* a maximum.

Deviation value *F,* the value *a* , with respect to which *z* Θ ισ a maximum.

However, I will treat these values not in the previous order, but according to the sequence *A* , *C* , *D* , *R* , *T* , *F.*

With the exception of *A* , the previous values, like the values of the previous chapter, are capable of a raw and sharp determination, whereas in *A they* are indistinguishable. The same small distribution table will here and there serve to explain the derivation, and the terms used here will be understood in the sense given in §§ 9 and 10. Bez. *A* go here *m* , , *m ',* in *μ* , ,*μ* ' and Θ , , Θ 'in Δ , , Δ ' on.

§ 77. Arithmetic mean *A* .

The arithmetic mean of a series of values *a* combines the following three properties:

1.That it the property itself, after which it is defined the ratio of the sum of *a* through their number *m* is *,* therefore:

(1)

or, insofar as Σ *a* by summing the *za* to win, = *az: m* ;

1.that the sum of the positive deviations Δ 'of it is equal to the sum of the negative Δ , according to absolute values, thus:

ΣΔ ' = ΣΔ , or ΣΔ ' - ΣΔ , = 0; (2)

3) that the sum of the squares of the deviations from it is less than any other value, in short:

$$\Sigma\Delta^2 = \Sigma\Delta'^2 + \Sigma\Delta,^2 = \text{minimum} \quad (3)$$

The former properties of *A* are so solidly connected that the others are given at the same time, and each of them can be derived with identical results, except that the derivation of the first property remains the most practical. Moreover, they are independent of a particular distributional law of *a,* and, beyond the collective theory of measure, they are considered not only for an ideal assumed to be infinite, but also for every finite series of *a* in arbitrary distribution.

The connection of the second and third sentences with the first given by the definition is found in this way.

Second sentence . Each positive deviation of A is $a' - A$, each negative for absolute values $A - a_,$, hereafter developed:

$$\Sigma\Delta' = (a' - A) + (a'' - A) + \cdots\cdots (4)$$

$$\Sigma\Delta_, = (A - a_,) + (A - a_{,,}) + \cdots\cdots$$

consequently, if μ' is the number of positive, $\mu_,$ that of negative deviations,

$$\Sigma\Delta' = \Sigma a' - \mu' A$$

$$\Sigma\Delta_, = \mu_, A - \Sigma a_,$$

$$\Sigma\Delta' - \Sigma\Delta_{,} = \Sigma a' + \Sigma a_{,} - (\mu' + \mu_{,}) A \quad (5)$$

or, because $\Sigma a' + \Sigma a_{,} = \Sigma a$ and $\mu' + \mu_{,} = m$,

$$\Sigma\Delta' - \Sigma\Delta_{,} = \Sigma a - m A, \quad (6)$$

and because $A = \Sigma a : m$

$$\Sigma\Delta' - \Sigma\Delta_{,} = \Sigma a - \Sigma a = 0 \quad (7)$$

Third sentence. Be the value, bez. whose $\Sigma\Delta^2$ is a minimum, initially set as unknown $= x$, so we have:

$$\Sigma\Delta^2 = (a' - x)^2 + (a'' - x)^2 + \cdots + (a_{,} - x)^2 + (a_{''} - x)^2 + \cdots \quad (8)$$

Although, it should unless we take the negative deviations for absolute values as positive, any negative deviation instead of $a_{,} - x$ etc rather $x - a_{,}$ are set etc; but $(a_{,} - x)^2$ is equal to $(x - a_{,})^2$, which allows to develop the previous value of $\Sigma\Delta^2$ in the given way. Now we obtain the minimum value of $\Sigma\Delta^2$ by setting the differential of its expression bez. x is zero; this gives:

$$2\,[(a' - x) + (a'' - x) + \cdots + (a_{,} - x) + (a_{''} - x) + \cdots]\,dx = 0 \quad (9)$$

thus by summing all a and $-x$

$$\Sigma a - mx = 0,$$

$$, \quad (10)$$

§ 78. If even the arithmetic mean for the collective measure of measure can not claim a predominantly greater interest than for the physical and astronomical measuring theory, then the combination of its three main qualities grants to it a mathematical interest in itself, and so much the more it grows because it establishes a relationship between the two teachings. In particular, he is at an advantage over *D* by the greater ease and simplicity of his exact determination; He is still surpassed in this by *C* , but the fact that the magnitude of the deviations simultaneously enters into the determination of the identity of his second property with the number gives him a more important interest than the *C.*Also, the following can be noticed. If one divides any series of *a* according to the random order, as contained in the original list, into a given number of fractions equal to *a* , and out of each of them determines the *A* , then the arithmetic mean of this *A* agrees with the general mean the whole series of *a* match. If, however, one proceeds according to the determination of *C,* then neither the central value nor the mean of the various special *C* generally agrees with the *C* obtained from the totality of *a* . If you proceed accordingly with the*D,* then the *D,* but not the mean of the special *D*coincides with the *D of* the totality of *a* .

Finally, the determination of *A has the* following practical advantage. If you have the *A of* a K.-G. determined from a distribution board with not too small *m* , so you will not only the total size "Gr." of the object for this table, by multiplying the *A* by the *m,* but also by probability, the total size of the object for each larger or smaller *m* obtained by multiplying that first determined *A* by the new *m* , but with an even greater probable error smaller is the *m* , and the farther the *m,*which one concludes, deviates from it. Conversely, the number of copies *m*belonging to it will be given a given total size Gr. to give probability, by putting m = Gr. **:** *A* ; since yes $\sum a = mA$ = Gr . *,* hence m = Gr. **:** *A.*

These sentences can z. For example, it may be useful to determine the space that holds a given number of people of randomly varying size. Neither the central value nor the densest value allow appropriate use.

§ 79. It may be that from the *A* different K.-G. or also the specially determined *A of* different divisions of the same K.-G. to draw a common means, and, if these *A are obtained* from different *m* , has to distinguish whether the definitive means should be drawn without or in consideration of the difference of the *m* . Let A_1, A_2, A_3 ... *be* special means, respectively from m_1, m_2, m_3 ... Drawn moderately. Regardless of the difference of the *m* , the mean of the *A in* question will be:

, (11)

where *N is* the number of *A;* but considering the diversity of the *m* , it will be:

(12)

and agree with the mean that is obtained by dividing the total of all a by the sum of all m .

The first means the singular, the latter the summary. Depending on the nature of the task, one or the other kind of education may be preferable. Put the means of the body length of the Chinese, Negroes, Malays, Americans and Europeans Caucasian race to be determined; but of the Europeans there are 1000 measurements, of each of the other races only 10-20 measurements; so the second, the summary form of fund-raising would be inadmissible; for, as easy to see, would the average body length of these different races because of the disproportionately vast weight, which the Europeans by their great m obtained almost entirely agree with the Europeans, and indeed the definitive means mainly, by the special agents with the largest m determine what contradicts the nature of the task. Here only the first, the singular kind of middle-education is useful; and that not all *meters* are the same size, only reduces the security of the provision against the case that the set of m is equal between all A distributed. In general, disparate objects (see § 14) will give more reason for the first than second means; whereas the special A from various departments of a unanimous object are to be combined according to the principle of the second means.

It can also be that one. instead of having to draw an arithmetic mean of different C or D from different A , then the corresponding distinction between singular and summary means applies, and the same points of preference for one or the other apply.

§ 80. Central value *C.*

In contrast to the three main properties of the arithmetic mean A , the central value C combines the following three main properties:

1. The property, given by its definition, of having as much greater a' about itself than a smaller $a_{,}$ among itself.

2. The property of having equally many positive and negative deviations from itself, so that $m' = m_{,} = \frac{1}{2} m$.

3. The property that the sum of the positive and negative deviations from it is smaller by absolute values than by any other value, hence bez. same $\sum\Theta$ ισ a minimum.

These properties, too, are in solidarity with each other, and for any given series of a , they are ruthless to a particular law of distribution, as is true of the three principal qualities of A.

The inference of the second property from the first is self-evident and needs no explanation. The connection of the third with it, however, follows in this way.

If the value of the third property is initially set as unknown $= x$, then the sum of the deviations with respect to x according to absolute values must be stated as follows:

$$\sum\Theta = (a' - x) + (a'' - x) + \cdots\cdots + (x - a_{,}) + (x - a_{,,}) + \cdots\cdots \quad (13)$$

To get the minimum of this sum, we have the differential of the same bez. *set x* equal to 0; which gives:

$$- m'dx + m\ ,\quad dx = 0\ ,\ (14)$$

therefore:

$$m\ ' = m\ ,\ (15)$$

which corresponds to the concept of the central value.

I have this property of the central value first in a treatise [1] proven over them and respond by generalization of the path that leads to drawn general conclusions, but to which I have no reason here.

[1] [About the initial value of the smallest variance, its determination, use and generalization; Treatises of the math.-phys. Class of Kgl. Sächs. Gesellsch. d. Scientific. XI. Tape; 1878. p. L - 76.]

§ 81. The following can be added to the central value for collective measurement.

If one were to think of all copies of a K.-G. put into a big urn, for which one can look at the world itself, and randomly pulled out a specimen, the likelihood would be to pull out a larger and a smaller specimen than *C* , and on very many trains would really preserve this same probability whereas with respect to values greater than *C,* the probability of extracting a smaller object, with respect to values smaller than *C,* outweighs the likelihood of extracting a larger copy. After this you can *C*in the same sense the probable value of a K.-G. to call what one calls the probable error of a means of observation, insofar as the probability of its exceeding and falling short is the same.

In the customary way of arranging the distribution board of KG, namely recruiting plates, in such a way that of the specimens which go below and above a certain size limit, only the number, not the size, is indicated, the possibility of an exact arithmetic mean being omitted pull; and then, instead of this, the central value, which can be determined just according to the mere number, can be compared with these comparisons, for example. For example, between different years and places from which the measurements originate, a procedure which has served me to work long years of Belgian recruits from the various provinces of Belgium, to note the gait and parallelism of these measures through time and space.

§ 82. The derivation of the *C* from a series of values *a* , which are ordered by their size, has to be done in principle by counting from each end in succession the middle to the same number of values and the value or intermediate value between two values *C* , in which both counts coincide, provided that the notion of *C is* clearly sufficient for both sides to have the same number of deviations, and thus equally many different values above and below each other. But there are two cases to be distinguished, firstly, where the *a* , which one comes upon in this double counting, or the two *a* between which the result of the counting arrives, are simple, or where, as is generally the case with our distribution tables, they have a $z > 1$.

Let's first look at the first simpler case. For the first sight now above rule appears to boil down here, that if the number of values *m* is ½*m* values, be it counts down by one or the other side, and the ½*m*- th value as *C* increases. In the meantime, it is easily convincing that this enumeration leads to a different value as it happens from one side or the other. Because be z. For example, the following series of four values:

a, b, c, d

given, one would find the ½ *m* -th, ie the 2nd value from left in = *b,* from right in = *c* . Or, instead of a straight one, we take an odd *m,* z. B. 5, by setting up the following series:

a , *b, c, d* **,** *e* ,

Thus one would find the 2½th value from the left in between *b* and *c, and* from the right in between *c* and *d* , whereas *c* only corresponds to the basic rule, and to one side just as many larger values above each other than to the other below to have. On the other hand, one satisfies the requirement of coming from the same side to the same *C* , even and odd *m* , by taking the ½ (*m* + 1) th value (ie the mean between ½ *m* and ½ *m* + 1) takes. In fact, in our example with the even *m* =4 one comes to a value between *b* and *c* from one side to the other , in the example with odd *m* = 5 both values *c* .

If, however, we now take the second case, which is actually of only interest, which occurs in our distribution tables, that the count arrives on either side from an *a* , or arrives between two *a* , which are affected by a *z* > 1, then we would turn to roer determination by us this *for* all the concerned *a* think falling themselves, and the *C* first case with that of *a* self-coincidentally or second case between those two *a*fall, and in the absence of certain stops, as a means of And so in our example table (§ 68), 11 should be regarded as the central value , in that if we count the previous rule by ½ · 81 = 40½ from both sides, these arrive within the *z* = 30 ascribed to *a* = 11 .

But in order to obtain a sharper definition, we must take into account that the *z* = 30 specimens are distributed throughout the interval from 10 to 12, and in consideration of this, take an interpolation of this interval taken as *I* to a matching *C* by counting from both sides not from ½ (*m* + 1) *,* but from ½ *m* specimens, as appeared most natural from the outset. In fact, in order from the top down to the 40th (on the location of the table) (di ½ *m* th) to enter values, we have to take into account (which is directly in the column of *S* **,** read) that until the end of the preceding interval, and thus to the beginning of the *I,* 18 copies suffice; missing to fulfill the 40 still 22 copies, which overlap in the interval *I.* Now we conclude: how these interdisciplinary 22 behave to the total number 30 of the *I* , so the value *x* to be added to the beginning of the *I, that is* to say 10 *,* so-called intervention in *I,* to the size of *I, that is* to 2, thus:

$$22 : 30 = x : 2,$$

di

$$x = 44 / 30 = 1.467$$

$$C = 10 + 1.467 = 11.467.$$

Now, if we start with the counting from the bottom upwards, then 32 copies are enough to reach the interval I , thus missing 40 or 8, which fall into the interval I itself, namely the part $I - x$ *of* which, from x to the second limit of the *I, which* ranges up to 12. Now we close again:

$$I - x : I = 8 : 30.$$

Since $I = 2$, one has

$$30(2 - x) = 16,$$

from which the increase x to the first limit 10 is determined as above = 1.467, which leads to $C = 11.467$.

Since the second *mode* of determination up to ½*m* from below leads to the same result as the first one, but this one is simpler, we can be satisfied with the determination of C , and obtain the following formula [2] for the determination of C :

, (16)

where g_1 is, as before, the initial value or the first limit of the interval to be interpolated, z_0 is the z *of* this interval, y is the number engagement in the same, ie the number by which the preposition v *must* still be increased by ½ *m* .

2) If, instead of the simple interpolation, the sharper, using second differences, occur, then $x = C - g_1$ would have to be solved by solving the equation (16) of Chap. IX instead of as above obtained by solving equation (13) of the same chapter.]

§ 83. Tightest value *D.*

If we define the closest value first as short as that which occurs most frequently under a series of a , or on which the largest z falls, then such a value can not be derived from any given series of a like the two previous principal values, and indeed has only for the collective gauge a relevant, but for them very important importance [3] . In fact, we put z. For example, you can place the following series of five a at random :

1, 3, 4, 6, 16,

so, as an arithmetic mean, we will have $A = \sum a : m = 30 : 5 = 6$; as the central value (by coinciding the count of right and left) $C = 4$. But what value should we take as the densest value, since each value occurs only once, so that all *z are* only

1. Other series can be set up arbitrarily in which, although different *, for* at various *a* case that the same maximum *z* but at more than *a* repeated what not to decide which of the *D* to display. But in distribution boards by K.-G. with big *m*Those who satisfy the requisite for a successful investigation either do not even have such cases, or are tolerated if primary plates are used, examples of which are given in Chap. VII, by eliminating the necessary reduction in the way that the maximum *z*falls only on one of the reduced *a* . However, it must not be forgotten that with the fact that we all maximum *for* the reduced *a,* where it is written beige refers; we obtain only a crude determination of the densest value, which is only more or less approximate to the ideal one, given the assumption of an infinitely large *m* at infinitely small *I* would have to, and must seek to approach as possible in a later manner. In general, one can only say that this value is to be found within the interval substituted in the interval table for the reduced *a* as its perimeter interval.

3) Of course, the assumption, which has not been criticized up to now, that the observation errors in non-random observations should be symmetrical. of the arithmetic means of observation, the great importance of the *D would* also extend to the physical and astronomical theory of measurement. [See Chap. XXVIII.]

That at symmetrical W. the deviations bez. *A* the densest value *D* essentially coincides with *A* and *C* is mentioned several times; after the generalization of the GG for the asymmetric W. of the K.-G. but, in general, he deviates from it, and then possesses none of the three fundamental qualities, whether of *A* or *C;*on the other hand, the qualities enumerated in § 33, of which the most important ones are: 1) that he is the densest in the sense given, 2) that the law of proportion, and 3) that the two-fold GG is the same, on which further depends, that in order to obtain a simple distribution law for collective deviations, the deviations have to be made dependent on it rather than on *A* or *C.* It may be added that *D is* the most probable value of a K.-G. from the following points of view represents.

If one takes from the totality of *a of* a K.-G. a copy by Random out, the value is *D* more likely than others to be taken each, and the nearby him *a* with a, his, nearly equal to coming, but verschiedenenW., depending on them to one or the other side of *D* fall.

Hereafter the importance of *D* for K.-G. from more than one point of view, that of any other principal value, but without obstructing the fact that these remain noteworthy according to the qualities which he does not share with them, and to the complete characterization of a K.-G. belong; He also stands in the disadvantages against all the others in that its as accurate as possible representation is cumbersome and requires a work of calculation, which does not need for others. This would now be discussed in more detail; but I would rather spare the rather laborious discussions of its derivation on a particular chapter, to discuss the following three main values.

Section 84. Dividing value *R*.

The value which has an equal sum of *a* over itself as itself, and which therefore has to form the dividing line between the smaller and larger *a* ordered according to its size , if by summation of the smaller *a the* same total quantity is to be produced as by summation the larger *a.*

[He lies above *C.* Because the number of above and below *C* located *a* is both appropriate, the concepts of the *C* according equal ½*m;* it is therefore:

.

so that a tie of the bottom only for a value greater than the upper sum of *C* can be reached. It is thus at the same time above *A* or above *D;* depending *A* or *D* is less than *C* is, whereas it may be below one or the other of these two main values may, if one or the other is greater than *C* is. However, its location first with regard to the generally as already known presupposed *A* to determine Assume that *R* above *A* lie.]

Now λετ $\sum a$, ' $\sum a'$ *be* the sums below and above *R,* $\sum a''$ and $\sum a''$ the sums below and above *A,* then count $\sigma = ½ (\sum a'' - \sum a'')$ up, ie to the larger values of to *A* from to *R* to arrive.

Proof . After view of the line scheme

——————|——————|

A R

is the lower sum of the *a* bez. *R* equal to the lower sum bez. *A* plus the sum between *A* and *R,* which is called σ , ie

$$\sum a_{,} = \sum a'' + \sigma .$$

The upper sum bez. *R* is the same:

$$\sum a' = \sum a'' - \sigma ,$$

well there

$$\sum a_{,} = \sum a' , \sum a'' + \sigma = \sum a'' - \sigma ,$$

, (17)

There

$$\sum a'' = \mu_{,} A - \sum \Delta_{,}$$

$$\sum a'' = \mu' A + \sum \Delta' ,$$

so you also have:

(18)

These directions are ruthless to a special law of distribution, only that a raw and sharp determination can be distinguished in the usual way. [They retain their validity even in the event that *A is* above *R* ; Ηοωεϖερ, σ then becomes negative and, therefore, taken from its absolute value, it is to count down di after the smaller values of *A* to arrive at R.]

In our illustrative example, $A = 11.4$; $\sum a'' = 369$; $\sum a'' = 543$, hence our present? = 87; We have to count this sum from 11.4 upwards, ie to the larger *a* to, to get to *R* and to interpolate the interval 10 - 12 with *za* = 330, which leads to $2 \cdot 87 : 330$ = Add 0.527 to 11.4; gives *R* = 11,927. [But if one sets $\sum 'a''$ = 362,7; $\sum 'a''$ as before (§ 72)= 549.3, hence σ= 93.3, it is logical to find the difference $R - A = x$ from the equation: $93.3 = (11.4 + \frac{1}{2}x) \cdot 15\, x$ with the value 0.533; gives with the above values substantially matching *R* = 11,933.]

[Instead of determining *R* as a function of *A* , as is done here , it can be found just as much as *C* or *D* ; then, of course, $\sum a''$, $\sum a''$ and, accordingly, the deviation numbers and the variance sum. *C* or *D* instead of bez. *A* take. At the end of *C* we obtain the following definition: $\sigma = \frac{1}{2} \sum \Theta$ (ορ *C*) *;* on the other hand, at the exit of *D* : $\sigma = \frac{1}{2} (m' - m_,) D + \frac{1}{2} \sum \partial$. Moreover,*R* can also befound directly, without reference to a predetermined other main value. This is done by looking upthe interval in which*R* comes to lieby adding the*a*from both ends of the distributiontable, and then determining in this engagement intervalthe mesh sum*Y of*the kind that the bias sum increases by the mesh sum equal to half the sum total*a*is. This leads, using the terms defined in § 69, to the formula:

(19a)

or to

, (19b)

depending on the determinations made in accordance with § 72, the intervention measure *x* , ie the value by which *R* exceeds the lower limit g_1 of the interval *I* , according to the proportion

$$x : I = Y : a_0 z_0$$

or more exactly according to the equation:

calculated and g_1 is added.]

[Finally, it deserves to be mentioned that the position of *R* in other ways than that of *A, C,* and *D* depends on the *a of* the distribution table. If one multiplies each *a* by one and the same amount, then *A, C* and *D also* increase by the same amount, so that the position within the table is preserved; however, the specified increase causes an approximation of *R* to *C* of the type that, in unlimited proliferation *R* with *C* coincides. This follows directly from the fact that those between *C* and *R* The preferred sum of *a* , di σ *,* constantly equal to ½ AΘ (Ref.*C)* and thus in increasing *a* distribution of a ever smaller interval.]

§ 85. The heaviest value T.

Each value *a* is a suitable to our studies, distribution, and generally speaking, depending on its size and *for* how often it occurs, a different product *za* , and it is now after *a* question for which this product is a maximum. First, it can be remembered that it coincides with the densest of values. But with this it depends only on the size of the *z* , not the *za* . There are values *a* that are larger than *D* , and although they are rarer than *D* , they give you the size of the *a* concerning the *za* up to certain limitsWhat they deliver, an advantage.

In any case, *T can* only be off to the positive side of *D* because, as the values *a go* below *D,* both *a* and *z* decrease. After a rough determination, in our example *table T* with *D would*at the same time fall to *a* = 11, as long as the maximum *za* = 330 is found. However, according to strict definition, both are mutually exclusive, and if the two-sided GG is presumed to be true, one has to use the following formula at all:

, (20)

From our example table § 68 we find the proportioning procedure to be discussed in the next chapter

$D = 11.6$; $e' = 1.9$;

hereafter

$T = 12.1$.

Now one may ask, what is the empirical meaning that the maximum of *z* falls to the value of *T* thus determined . In this respect, one has to remember that after a sharp viewing each *a*actually a distribution table for a whole interval of the size of the *i* represents this table, of which the respective *A* is the center. Thus, with the value *T* = 12.1 for our distribution table, where *i* = 2, it is said that, among all intervals of this panel of size 2, the interval whose center is *T* = 12.1 is the interval 11.1 - 13.1 a larger *za* contains, as any other interval of size 2.

[But this is not confirmed; because the *za of* the interval 11.1 - 13.1 is equal to 296, while the *za of* the interval 10 - 12 is equal to 330. However, this does not prove the incorrectness of the above theoretical mode of determination of *T* , but merely suggests that the theoretically required position of the heaviest value does not coincide exactly with its position empirically presented in the table, which, incidentally, is not to be expected from the outset. That this is also true of the tables of empirically given K.-G. is not significantly different, as shown in the following example.]

The distribution board for the vertical circumference of the skull with $i = 5$ mm (§ 58) gives the determination of the *D by* means of the proportioning method:

$$D = 409.7;\ T = 410.1;$$

then here on the interval 407,6 - 412,6 the largest *za* falls. Whether this can really be found can be empirically tested on the distribution table, and for comparison let us choose the interval of the densest value 409.7, ie, after appropriate determination, 407.2 - 412.2.

Since the *za of* the intervals in question are not immediately given in the distribution table , because these intervals are not even given with their *za* , but rather the interval of the heaviest value, as well as that of the densest value, overlaps between two intervals of the given table, so must calculate interpolationsmäßig what proportion to the searched *za* delivers each of the two intervals, and by summing these shares both the *descending* of the interval, what *D* than what *T* has set, find out what I do not want to detail here [4)]. After that I found the *za* for the above examplethe densest value 26631, that of the *T* equal to 26656, so, as might be expected, the latter very little, but, as we would like, somewhat larger than the former. [But nevertheless, the *T* determined theoretically from (20) is different from that empirically to be taken from the table; because for $a = 413$, the even larger value $za = 26845$ results.]

[4)] [In the present case, this calculation is simplified as a result of the $z = 65$ common for $a = 408$ and $a = 413$, and the *za* for *D* resp. *T* equals 65. *D* resp. 65th *d* .]

Proof. Since *T is* greater than *D* , we set

$$T = D + \partial, \quad (21)$$

where ∂ a positive deviation of *D* , and determine ∂ *,* as we

$$za = z(D + \partial) \quad (22)$$

set this value to obtain a maximum equation with respect to ∂ , differentiate and set the differential equal to zero, where we for simplicity, the dashes above to *z, a,* ∂*, e* omit that are actually attaching to the location of these values above *D* to denote ,

So we have:

, (23)

of which the last value is z . To find now , z must be expressed *as* a function of ∂ , which can be done by assuming the probability ρατιοσ for D on the positive side of D after the two-column GG . Ηερειναφτερ, ασ is well known, the probability φ∂ οφ a value ∂

, (24)

where $h = 1{:}\ e$. However, in the normal case of a large m , φ∂ χαν also be εξπρεσσεδ by z: $\boldsymbol{m}'$, hence

, (25)

From which follows:

(26)

and because

, (27)

so:

, (28)

where z is omitted *as* a common factor, and, by reversing the signs and considering that $h = 1 : e$, *one* obtains the following quadratic equation:

$$2\,\partial^2 + 2\,D\,\partial - \pi\, e^2 = 0, \quad (29)$$

from which ∂ can be determined.

This gives first:

, (30a)

of which only the upper sign is useful; or:

(30b)

and:

(31)

§ 86. Deviation weight *F.*

One can still speak of a characteristic deviation value , which is analogous to the heaviest *a* value and is to be calculated analogously, after which the heaviest deviation value can be called. There you asked, to which *a* coat of the largest *za* , here you ask to which Θ falls the largest *z* Θ *,* and provided at the output of a given main values *H* with Θ at the same time *a*= *H* ± Θ is given to which *a* coat of the largest z Θ , a value by no means the heaviest *a*Values coincide. Meanwhile, the analogy fails in the following points. The maximum of *za* is independent of the principal value, which one prefers to prefer, since this does not change the factual values of *a* and related *z* , except that a simple calculation of the largest *za is possible* only at the output of *D* according to our general distribution law is. In contrast, the value *z* Θ depends on the principal value from which the deviations are to be calculated, since the values Θ themselves depend on their size. However, it remains with the calculation of the heaviest *a*Value is equal, that even with the heaviest Θ– value, it can only be done at the end of *D* on the basis of our general distribution law, and the application of the result can be disturbed by a lack of fulfillment of the props. Finally, the analogy does not hold in the sense that there can normally only be a maximum of *za '* in every distribution table; whereas for each side of the chosen main value there is a special maximum of z Θ resp. of z ' Θ ' and *z* , Θ , short *F '* and *F* , gives, which is just at the output of *D*is subject to a corresponding calculation.

For explanation, we take the reduced table for the vertical circumference of the skull (§ 58) with *E* , = 368; *i* = 5, for which according to § 61:

D = 409.7; = 14.9; = 13.0;

Values that will be considered in the calculation; and we form according to the *a* and deviations of the *a* from *D,* d . i. ∂ *,* in that table, the following table of related values:

D = 409.7; = 14.9; = 13.0.

a ,	∂ ,	*z* ,	*z* , ∂ ,
383	26.7	17	454
388	21.7	24	521
393	16.7	36	601
398	11.7	41	480

403	6.7	59	395
405.5 - D	0 - 4,2	55	115
a '	∂ '	z '	z ' ∂ '
D - 410.5	0 - 0.8	10	4
413	3.3	65	214
418	8.3	51	423
423	13.3	40	532
428	18.3	17	311

It can be seen here that ∂ and z take a reverse course in so far as ∂ decreases *as* each of its a approaches D on each side, z grows; conversely, upon removal of a of D. If z and ∂ obey an inverse relationship, z ∂ would remain constant throughout the whole series of values, but this is by no means the case, as can be seen from the last column, according to which the a , side is a maximum of z , ∂ , just F, , At ∂ , = 16.7, and a , = D - ∂ , = 393; and on the a' side a maximum of $z' \partial'$, short F', takes place at $\partial' = 13.3$ and $a' = D + \partial' = 423$. [The same values also mark the maxima of the $z \partial$ by sharp determination by simple interpolation .]

As can be seen, the empirically determined maximum value of z , ∂ , = F , is very close to the above given value e , = 14,9 and the empirically found maximum value of $z' \partial' = F'$ on the a' - Side very close to the values given above e' = 13.0; and, in fact, the result of the calculation to be subsequently established, on the basis of the validity of our law of distribution, is that

; (32)

[But if we determine the interpolation values z , ∂ , and $z' \partial'$ corresponding to the values ∂ , = 14,9, and $\partial' = 13,0$, taking into account that $i = 5$, one finds z , ∂ , = 563; $z' \partial'$=529, whose comparison with the true maximum values of the panel reveals the degree of correspondence between the theoretically required and empirically presented values.]

[Proof. Based on the valid as valid two-column GG:

, (33)

where $h' = 1 : e'$, the value for obtaining the maximum equation for $z' \partial'$ is:

(34)

with respect to ∂' and set the differential equal to zero. You get like this:

, (35)

that is, since the coefficient of $(1 - 2h'^2 \partial'^2)$ can not disappear in its nature,

or . (36)

In the same way follows for the lower deviations:

, (37)

But now e' and $e_,$ the mutual mean square deviations, are such that the theoretical meaning of the deviations *F'* and $F_,$ with respect to *D,* is to represent the mean square deviation of the upper and lower values.]

XI. The densest value.

§ 87. [Since the densest value is the initial value of K.-G. a fundamental position in the collective theory of doctrine, it is necessary to discuss its mathematical meaning and its mathematical determination to be founded on the latter. Here it is essential to divide the empirically denominated value denoted by D_i, given by the table, from the theoretically most probable value, designated as D_p, required by the law of distribution, and to treat each separately.]

[The existence of D_i is based on the fact that the *z of* the plate used for a K.-G. indicate the number of copies of the size *a* , are not consistently constant, but rise and fall. As long as now in raw determination *for* directly as the beige signed *a* conceived associated and thus the measured between the *a* of the panel covered *a* values are not considered occurring and can only with the greatest *z* -prone *a* themselves are claimed as the densest value ; and there is then no means, in the event that several consecutive *a* to have the same maximum *z* , to raise the doubt, which *a* in fact represents the densest value [1)]. But it is taken into account that the intervals between the measured *a and* the relatively small number of the measured specimens and the inaccuracy of the measurement owe their existence, whereas the unlimited number of copies of the K.-G. is distributed without interruption to all a lying between the extremes *a* , so one has to look in the given table values only the document on which a functional connection between the *z* and the *a* builds up. When made the same, the closest value simply results as the maximum of the function being constructed.]

1) [The presence of two mutually equal, separated by intermediate values of maximum *z* is not to be considered as these cause the appearance of two different values and densely so a research Mi-disparate K.-G. to which the distribution laws no direct Find application would show.]

[In the production of this functional relationship, it is important to note that - which is already due to the imprecision of the measurement and the consequent existence of the primary intervals - the *z of* the table is not a single value of the sought function but a sum value; to refer to the associated intervals, and thus as integral values, taken for the limits of the intervals to apply. Moreover, the principles of the interpolation calculus apply, which implies , within a certain range , the number of copies of the size *a,* which is generally denoted by ζ , as a whole rational function of *a*presuppose and then, by means of the given *z of* the table, to determine its coefficients such that the sums of the ζ , d. i, their integrals between the boundaries of the considered intervals agree with the given *z of* the panel for the same intervals; The number of consecutive intervals to be taken into account depends on the degree of the function assumed or on the number of coefficients to be determined, and the degree of precision attained increases with the increase of that number.]

[If so provided that for the range of a value *a,* which in the interval to the center of a_0 and a *z* equal to z_0 lie, ζ is either constant or by a linear function of *a* depicted, or by those of the second degree in the first case, only the z_0 of the interval itself, in the second case the *z of* one of the two neighboring intervals, and in the third case the *z* of the two neighboring intervals, will be used to determine the constants. One finds so, if the *z*of the interval after the upper extreme with z_1 , which is designated in the opposite direction as z_{-1} , and which is called the interval size asserted in the extension of the whole table *i* , in the first case:

; (1)

in the second case:

or = ; (2)

in the third case:

; (3)

Formulas whose validity range extends in any case over the interval with the boundaries $a_0 - ½\, i$ and $a_0 + ½ i$.]

[If we now want to determine the densest *a of* the interval on the basis of the functional dependence thus constructed , then formula (3) proves useful; because (1) delivers consistently constant, (2) constantly increasing or constantly decreasing values. From (3), however, the maximum value or closest value results:

, (4)

if only $2 z_0 - z_1 - z_{-1} > 0$. If the latter value is smaller than zero, provides *a* minimum is, but is $2 z_0 - z_1 - z_{-1} = 0$, (3) is linear, and for determining a maximum unusable. If, moreover, the maximum must lie within the interval under investigation, then both z_1 and z_{-1}, each individually, must be less than z_0.]

[Instead of the middle a_0, the determination of the densest value can also refer to the limits of the interval: $g_1 = a_0 - ½i$ and $g_2 = a_0 + ½i$. One finds, if $a - g_1 = x$ *is* set:

; (5)

what makes the simple proportion:

$$x : (i - x) = (z_0 - z_{-1}): (z_0 - z_1) \quad (6)$$

follows.]

[The determination of D_i is thus accomplished by means of the above formulas, by first looking up the interval with the maximum z , ie the most densely denominated value, and then the position of D_i within this interval by the proportion of the proportion (6). or calculated from equations (5) or (4). If only a maximum z exists , the achieved accuracy is sufficient, and the use of sharper interpolation formulas considering the z of four or more neighboring intervals is generally unnecessary. Yes, one still gains a useful determination even if two adjacent maximum $z$leave the raw determination of the densest value in the dark. Namely, when $z_0 = z_{-1}$, $x = 0$, and when $z_0 = z_1$, $x = i$, so that always the common boundary of the two, with the maximum -z affected intervals as D_i in claim to take is.]

§ 88. [In this way the values D_i *of* the various reduction stages and reduction positions of VIII. calculated. It will not be otherwise in the later chapters. However, it may be desirable to have a sharper formula in case two adjacent maximum -z occur. Yes, it would be such an imperative if - which is hardly to be expected and, when necessary, can be avoided by changing the reduction location - three succedierende maximum *for* the failure of the above formulas would require. Then there is another interval to add to the previously considered to ζ to be able to determine as a function of third degree. Let this be the interval with $z = z_2$ following the interval with $z = z_1$. If, as above, $a = g_1 + x$ or $= g_2 - (i - x)$, where g_1 and g_2 are the lower and upper limits of the interval with the center a_0 and $z = z_0$, we obtain :

$$\zeta = \alpha + \beta (i - x) - \gamma (i - x)^2 - \delta (i - x)^3;$$
$$12 i \alpha = 7 z_0 + 7 z_1 - z_{-1} - z_2; \; 12 i^2 \beta = 15 z_0 - I5 z_1 - z_{-1} + z_2$$
$$4 i^3 \gamma = z_0 + z_1 - z_{-1} - z_2; \; 6 i^4 \delta = 3 z_0 - 3 z_1 - z_{-1} + z_2 . \quad (7)$$

It follows as a maximum value when z. Eg $z\,0 = z\,1$ and $z\,0 > z\,2 > z\,\_\,1$:

, (8th)

One also finds:

if $z_2 = z_1 = z_0$;

if $z_{-1} = z_1 = z_0$ (9)

according to which the position of D_i changes, depending on the three maximum - z following or the preceding interval. This uncertainty can only be counteracted by taking into account the two neighboring intervals.]

[This is done by $z_0 = z_1 = z_{-1}$ accepts and except for the following interval with $z = z_2$ have the previous interval with $z = z_{-2}$ taken into account, we obtain for the determination of the maximum, for $x = a - g_1$, the equation:

$$\alpha + 2 \beta x + 3 \gamma x 2 + 4 \delta x^3 = 0;$$
$$12 i^2 \alpha = - z_0 + z_{-2}; \; 8 i3 \beta = z_{-2} - z_2; \; 6 i^4 \gamma = z_0 - z_{-2}; \quad (10)$$
$$24 i^5 \delta = - 2 z_0 + z_2 + z_{-2};$$

under the condition:

$$2 \beta + 6 \gamma x + 12 \delta x^2 <0.]$$

§ 89. [While the existence of D_i is independent of the existence of a distribution law, and its determination can be achieved in successive approximation by interpolation, the existence of D_p is governed by the presupposed distribution law, our case by the two-sided GG, and its calculation from the given tabular values is to be made on the basis of its mathematically formulated properties. It would indeed, if the unavoidable, unbalanced contingencies would not hinder an exact application of the law of distribution, the densest value from the outset possess the properties of D_p, thus $D_i = D_{be\,p}$; and it would then no reason exists, in addition to D_i still D_p to calculate, if not the strongly worded properties in this case D_p would offer greater security than the approximations of Interpolalionsverfahrens. Insofar as the course of the table values never fully corresponds to the requirements of the law, D_i and D_p differ; and it must be independent of D_i and D_p are determined in order to gain a measure of the applying of the distribution law in both the differences in their position, as well as in D_p to obtain a more suitable initial value as in D_i for the application of that law.]

[D_p, in solidarity with the two-sided GG, is now defined by the property that the numbers of the lower and upper deviations with respect to it behave like the mean values of the lower and upper deviations, or that:

$$m : m' = e, : e'. \quad (11)$$

Since this property of the theoretically most probable value is an outgrowth of the law of distribution, the validity of this law presupposes from the beginning that one and only such a value exists in our distribution tables and should be sought near D_i. But it has an interest to demonstrate that D_p the one hand, not as *A* or *C,* exists in any panel and on the other hand occur in several support can.]

[For this purpose, suppose a partitioning table with equidistant a, of which z is one constant throughout, and the other is consistently the same multiple of the associated a.]

[In the former case, the *z should be* distributed evenly over the whole table; it is therefore, between the limits $a = b$ and $a = c$:

$$\zeta = \alpha ,$$

where α ισ a constant; and for any a, one finds:

$$e, = ½ (a - b); e' = ½ (c - a)$$

$$m, = \alpha(a - b); m' = \alpha (c - a),$$

so that every a possesses the property of D_p.]

[In the second case, the continuous distribution results through interpolation:

$$\zeta = \alpha \cdot a$$

and choose as limits $a = 0$; $a = c$, one obtains with respect to any *a:*

; ;

; ;

so that as solutions of the equation:

$$e_, m' - e' m_, = 0$$

only the two values $a = 0$ and $a = c$ result, for which $e_,$ and $m_,$ resp. e' and m' **are** equal to zero. From these limits, however, the conditional equation for D_p *is* fulfilled from the outset in each panel , without being claimed as D_p values. There is thus no D_p within the panel in this case .]

[As a result of this occurrence, it may seem desirable to have a criterion for the presence of D_p . Such is easily provided by the following consideration. Is detectable for the beginning of the panel $e_, : m_, > e' : m'$, for the end of $e_, : m_, < e' : m'$, it must for an average value of $e_, : m_, = e' : m'$ be, since the ratio $e_, : m_,$ and $e' : m'$ as a result of the continuous distribution of the z on the individual intervals, constantly change with the position of the value to which they relate. Now, however, when *such* α the z of $E_,$, z ω that of E' is and the lower limit of the interval of $E_,$ with b , the upper limit of the interval of E' with c is referred to, for the beginning of the panel :

; ;

for the end of the board:

; ,

In any case, there exists a value D_p within the table if:

; is.] (12)

§ 90. [For the calculation of D_p , at first only the proportion (11) can serve, since it defines this value. On the basis of this proportion, however, the following properties of the value D_p can be detected, which can be used in the same way for a calculation:

1. The arithmetic mean of below D_p located a , di $\Sigma a_, : m_,$ augmented by the arithmetic mean of the above D_p lies a , di $\Sigma a' : m'$, is equal to the arithmetic mean of all a, augmented by D_p itself.

, (13)

2) The difference of the mean values from the lower and upper deviations of a with respect to D_p is equal to the difference between the value D_p itself and the arithmetic mean of a; thus:

$$e_, - e' = D_p - A. \quad (14)$$

The connection of the latter equation with (11) leads to the further determination:

, (15)

where $u = m' - m_,$. By adding and subtracting (14) and (15) one also obtains:

(16)

The proof of (13) is furnished by substitution of the values

; (17)

into the equation $e' m_, = e_, m'$ resulting from the proportion (11), the equation:

(18)

derived and in the same

;

is set. In fact, the following equation follows:

(19)

by dividing by m the formula (13). However, this formula is obtained as follows from it when $\sum a_, : m_,$ and $\sum a' : m'$ from (17) by D_p, and $e_,$ resp. e', directly the equation (14).]

§ 91. [For the mathematical determination of D_p, equation (13) now offers the most convenient approach. However, knowledge of the interval in which D_p falls is required since the properties of the sought value are based on the deviation numbers and deviation sums and do not permit an absolute determination as possible for A. It must therefore, where such knowledge, the z. B. by previous calculation of D_i is absent, tentatively made the approach for any interval, and, unless the correct interval has been struck accidentally, repeated for another interval, the result of the first failed bill, however, being the choice of interval in the repetition of the interval Try to influence. If the board does not offer any major anomalies, these attempts will only be the choice between adjacent intervals.]

Accordingly, if a certain interval whose center a_0 whose lower limit is g_1 and z *is* equal to z_0 is chosen as the intervention interval and is calculated for the same ***v*** , ***n*** , ***V*, N** , then, in the case of a raw determination, 13):

; (20a)

or:

; (20b)

depending on D_p smaller or larger than a_0 . Thus, the former is true if $a_0 - D_p < ½i$, the latter if $D_p - a_0 < ½i$.

For sharp determination but is of the approach:

(21)

where *Y is* the mesh sum, *y is* the mesh number. If you put here after Cape. IX, formula (8) and (13), when *x* indicates the engagement amount = $D_p - g_1$ [2)]:

; ;

this yields the following equation for $x = D_p - g_1$;

$$\alpha x 2 - \beta x + \gamma = 0;$$

;

; (22)

;

with the condition that x is positive and less than *I.*]

[2)] [If one wanted the simpler but less accurate formula (6) of Chap. IX, namely $Y = a_0 z_0 x : I$, then instead of (22) a third degree equation wouldresultfor x ; thus, the loss of accuracy would also result in a loss of computational convenience.]

[Since, however, this mode of determination is by no means convenient , let D_p be related to any principal value *H* lying in the same interval in order to obtain simpler equations due to the particular properties of the *H* chosen .]

[For this purpose, like the numbers and the sums of the above and below *H* located *a* by m' , m'', $\sum a''$, $\sum a''$ denotes, further $D_p - H = x'$ and the between D_p and *H* lying *a* their number being equal to y', their sum equal to Y', so that:

; ,

One then wins from the approach:

(23)

for $x' = D_p - H$ the equation **:**

$$\alpha' x'^2 - \beta' x' + \gamma' = 0;$$

;

; (24)

;

which for $H = g_1$ passes into (22). From this must result an x' which is either positive and less than $g_2 - H$ (where g_2 is the upper limit of the intervention interval), or negative and, in absolute terms, less than $H - g_1$.]

[This equation, if either the arithmetic mean *A* or the central value *C* or the divisor value *R* falls within the interval of D_p and is chosen as *H* , leads to the following determinations:

1.Let $H = A$; $x = {}_{Dp} - A$; then:

; (25)

where μ , and μ *'are* the deviation νυμβερσ , ΣΔ the total sum of the deviations. *A to* introduce.

1.Let $H = C$; $x = {}_{Dp} - C$; then results:

; (26)

where $\sum a''$ and $\sum a''$ to C relate.

3) Finally, let $H = R$; $x = D_p - R$; then results:

; (27)

where m'' and m' *αρε* to be taken with respect to *R*.]

[The scope of these determinations is further extended if, in the case where D_p and the principal value referred to in the invoice fall within adjacent intervals, a shift in the engagement interval or, in other words, the engagement interval of abutting parts composed of two neighboring intervals. The z_0 of this composite interval is then composed of the proportionally determined *z of* its parts, while the bez. of the principal value.]

§ 92. [Of these formulas, (26) will generally be preferable. For (27) refers to a little interesting main value, the exact calculation of which even after Ch. X (19b) requires the resolution of a second-degree equation; while (25) is at a disadvantage in that A is separated from D_p by C and thus, less frequently than C, will be D_p in the same interval. Furthermore, it is not to be regarded as a disadvantage that equation (26) requires the knowledge of the two values A and C, since, in addition to D_p, we always also have *A*and *C will*calculate.]

[It is therefore advisable to reduce the calculation of D_p according to (26) to the simplest possible form based on the knowledge of C and A].

For this purpose divide (26) by $\frac{1}{4} m^2 x$ and write the equation as follows:

(28)

Now put:

, so ,

so you get:

, (29)

thus giving a continued fraction representation for ξ, which converges rapidly since $2 z_0 (C - A) : (Im)$ represents small values for our panels.]

[The course of the bill is therefore to set up the way that due to

;

sequentially:

$$\xi_1 = \alpha - 1;$$

;

;

Etc.

and, when the calculation has come to a standstill, the value of $x = D_p - C$ *is* derived from the found values of ξ . At the same time then results in a simple way the value of

same .]

[From equation (26), moreover, it follows that the position law of the empirically determined principal values *A, C,* and D_p is fulfilled from the outset with the proportions that apply to our tables. If you bring that equation into the form:

.

it follows, if so

.

that $A - C$ and x , di $D_p - C$, *can* not be both positive and negative at the same time. It is therefore, since the specified condition is indeed fulfilled by the distribution boards,

$$\text{either } A > C > D_p \text{ or } A < C < D_p,$$

as the Legislation requires it.]

XII. reasons

that the essential asymmetry of the deviations with respect to the arithmetic mean and the validity of the asymmetric distribution law with respect to the densest value D in the sense of the generalized Gaussian law (Chapter V) is the general case.

§ 93. According to the differences between essential and insignificant determinations (§ 4), one may be inclined to distinguish even a material and insignificant (or accidental) asymmetry of deviations with respect to a principal value, such as the arithmetic mean or densest value. Let us begin by considering the arithmetic mean *A* in this respect . It is certain that even with symmetrical W. the deviations are. *A* by unbalanced coincidences a difference between the distance of the extremes E', E , from *A* and a difference *u* between the number of mutual deviations $\mu'$and μ , and thus one can ask for features that indicate a significant asymmetry. *A,* which does not depend on unbalanced contingencies, differs from an insignificant or accidental one that depends on it. Apart from the ones in Chap. II, general, slightly indeterminate features, which distinguish essential from unimportant determinations, can be based on the fact that the difference *u* between μ' and μ , which results from mere unbalanced randomness ,a probability determination and that the probable size of the same can be stated. Now, as this probable difference is transgressed, it becomes less likely that asymmetry is a merely accidental one, and there are even rules to determine the degree of improbability, without, of course, an absolute certainty being attainable; on which I refer to the remarks in § 31 (historically) and refer to the probability formulas of the XIV. chapter. And so, as a guiding point of view, one could put forward a prime probability of considering only such cases of asymmetry with respect to *A* as essential, and to seek a probation of the laws of essentially asymmetrical distribution for those with respect to *A*resulting probable value of *u* is significantly exceeded.

In fact, from the outset, I have understood the matter in the first place, but have convinced myself, as I have already remarked in § 32, that this conception, which at first seems so natural and even necessary, completely misses the correct point of view. It would be durable if the symmetrical W. of the deviations with respect to *A*It would be the generally presupposed case, and except, as one might presuppose from the beginning and is still presupposed by QUETELET, suffer exceptions which would have been especially sought out and calculated. It turns out differently, however, if in the sense of the already presumptive view the essential asymmetry is the general case, which of the innumerable degrees in which the asymmetry may occur, the one where it disappears, only as special, in all severity perhaps never occurring case contains.

§ 94. Then there is no fundamental difference between essential and non-essential asymmetry; all K.-G. must, indeed, must be treated with the presupposition of asymmetric W., with care only that at finite *m,* because of unbalanced contingencies, the magnitude and direction of the asymmetry may happen to deviate from that which would prove to be essential in infinite *m* ; and the pervasive reason for putting it this way is that even in cases where, according to the present probability formulas, the asymmetry with respect to A might possibly be only accidental, the laws of asymmetry cited in § 33 are confirmed in a universality unexpected to me.

However, I confess that it has seemed strange to me, and that a riddle can be found in it that with as little asymmetry as is often the case in K.-G. of VII. And VIII. in conflict with the inevitable contingencies due to the finite nature of m , yet the laws of asymmetry outlined above are confirmed to be of curious generality and approximation.

Take z. B. the skull dimensions. 450 specimens of European skulls give for the vertical circumference (at $i = 5$ mm E , = 368) 220 negative, 230 positive deviations from A_2 the same skulls for the horizontal extent under appropriate conditions even 226 negative, 224 positive deviations, differences that are far too insignificant, so as not to be overgrown by unbalanced contingencies; and yet these cases, as well as many others of the same order of difference, give not less good affirmations of the established laws of asymmetry than the examples of greater asymmetry, which I have hitherto been able to explain only in such a way that the various elements by whose circumstances the concerned ones are To apply laws affected by the unbalanced contingencies, to be changed in the same direction and almost by equal magnitudes or in the same proportion, so that, on the contrary, only the absolute magnitudes suffer as the legal differences or relations of the elements, which does not assert that this same or proportional change takes place exactly, but only so far as to leave the latitude left by the laws intact becomes. This view may be in need of a more thorough mathematical discussion; in any case, in the expectation of this, the fact remains that even the weakest degrees of asymmetry$A$ audited distribution laws of asymmetry still prove their validity and thus help themselves, the general public a more than merely incidental asymmetry to prove [1] .

[1]. [See the theoretical derivation of the asymmetrical law of distribution §136, according to which the principal values differ only by quantities of the order i or 1 , which are to be so small that their squares are i^2 or 1: m finite quantities may be neglected.]

But if such exists in the sense given for the K.-G., then the application of mathematical formulas of probability to distinguish essential and nonessential asymmetry is actually idle. Always would like to prove for objects of weak asymmetry that the asymmetry with respect to A might possibly only be coincidental; What happens when the factual investigation proves that they obey the laws of essential asymmetry; However, since these formulas retain a certain theoretical interest in our field, I will deal with them in the following chapters, without having any subsequent practical reason to base them.

§ 95. If I now summarize the reasons which have led us to admit, instead of a substantial symmetry, a substantial asymmetry with respect to *A* and a generalization of the GG in the sense of the laws cited in § 33, these are the following.

1) Since there are cases of so great at any rate *u: m* is, where one can not help by far the greater probability reasons, the presence of significant asymmetry with respect *A* permit, the general case can not in substantial symmetry mar. *A to be* sought; but, if at all, something general for K.-G. In this relation we shall have in essential asymmetry, of which essential symmetry and weak asymmetry occur as special cases.

2) If one and the same K.-G. subject to a comparative distribution calculation according to the two-column GAUSS distribution laws (§ 33), which apply to essential asymmetry, and to the simple GAUSS distribution laws (§ 24 flgd.) which apply to essential symmetry, the former distribution calculation is from the outset in the advantage of that they have the empirically different **m'** , **m**, bez. *D* exactly reproduces on both sides, whereas the latter for the empirically different μ', $\mu_,$ bez. *A is the* same value ½ (μ'+ μ ,) = ½m, which therefore has to be too large for one side as much too large for the empirical deviation number than for too small on the other. This advantage, calculated in the principle of the compared accounts, for the calculation after the generalization of the GG for asymmetry would not in itself hinder the fact that in the individual distribution determinations the **m'** φ ' and **m**, φ , (§ 27) are all the greater on the whole overwhelming disadvantages against the method of calculation according to the simple GG asserted; but as far as I have made comparisons, the opposite is the case.

3) The laws of essential asymmetry, which §33 for the case of a sufficiently large *m* and fulfillment of the in Kap. IV., And continue to find their theoretical justification, are generally confirmed in the present study material with such an approach to ideal demands, as can only be expected in the case of unbalanced contingencies, and at the same time prove their correctness Theory.

So it applies first of all with respect to the proportional law. According to the explanations given it is that with respect to the value to which the largest *z* falls short with respect to the closest value, the number of mutual deviations such as the size of their mean values, di **m**, *:* **m**' = *e* , : *e* In other words, the value with respect to which the relation holds must coincide with the densest value directly determined by its *z* -max. Now that we have a distribution panel by appropriate reduction to such a regular course of *z*brought that an investigation of its laws and ratios is possible, we will find the determined therefrom according to the condition value that is relative to the same **m**, *:* **m**' = *e* , *e'* behaving, as falling within the interval on which the largest *z* falls, as one can convince oneself, if one considers on the one hand the D_p specified in the tables of the elements, everywhere on that condition , and on the other hand the distribution- table brought to the form of the interval- table from which the derivation has been made. Center of the Cape. XI specified interpolation method but you can *D*in the interval in which it lies, determine even more precisely than if one seeks to determine it directly according to the size of its z , and then, of course, one can not find in the tables of the elements a further confirmation of the law of proportionality in respect of that listed therein densest value D_p really $m, : m' = e , : e'$ behaves as D_p even as the value with which this relationship exists is determined. However, exceptionally, this value may fall into the neighboring interval under the influence of strong unbalanced contingencies and, in the case of an unfavorable reduction position, instead of into the interval with the maximum z itself; but in general it suffices to change the position of reduction in order to bring it into the interval in question.

But further we see the sharp as possible given on the basis of that proportion values D_p a baseline for deviations which satisfy the two-column GG, with random perturbations but that indeed may be missing anywhere, but only those of the same order, as well as in the distribution the observation errors with respect to the arithmetic mean occur and are tolerated, as the BESSEL comparison charts [2] prove between observation and calculation.

[2] [foundations astronomiae, section II, p. 19. 20.]

As far as the law of position is concerned, according to which the central value C and the arithmetic mean A lie on the same side of the densest value in such a way that C falls between A and D_p , then with its consequences it will invariably be applied even to the weakest $u: m$ in the tables of the elements are confirmed, and might be inclined to find here the most striking proof of essential asymmetry, since in essential symmetry D_p , C , Aonly by unbalanced contingencies, and then in indefinite mutual situation, could differ from each other. But there is nothing to be done about it. For it can be proved that the law of positions is a necessary consequence of the law of proportionality 3) , and if D_p in the tables of the elements is determined by the law of proportionality, then of course the law of position must be confirmed with respect to it, without being able to prove that this value corresponds to the maximum z , which fundamentally can only be done by direct comparison.

[3] [Comp. the conclusion of the preceding chapter.]

In contrast to this, the $\pi-$ laws, which establish certain values for the distances between D_p , C, A , presuppose the validity of the two-columned GG, without this being a necessary consequence of the law of proportionality, and thus contribute insofar as they are experienced confirm with such approximation, as allow unbalanced contingencies, however, essential to prove the presence of substantial asymmetry, as far as such is in solidarity with the two-column GG.

Finally, the features of the tables of elements and related comparison tables between observed and computed distribution for the presence of substantial asymmetry stem from it: a) that the D_p determined by the proportional law is so close to the directly determined D_i coincides, as allow unbalanced contingencies; b) that the deviations from the exactly as possible in the former certain paths D_p satisfy the two-column GG in a satisfactory manner; c) that the π -Laws are met with sufficient approximation. Of course, the fulfillment of the props of Chap. IV, which may even lead to a successful investigation of the K.-G. must be fulfilled. Insofar as the given criteria generally apply under these conditions, a conclusion can be drawn on the general occurrence of essential asymmetry.

4) Do we understand related K.-G. pursuant to the following examples, there are quite a few cases where the *u* thereof in the available to it *m* is too small to allow not everybody especially the possibility of dependence on merely random asymmetry left in the direction but in all such coincidentally, or a modification of the objects so legally following as is not compatible with mere coincidence.

Thus, in recruiting measures of very different countries, as far as they are to be regarded as complete, I have always found the asymmetry with respect to *A* positive, with daily and monthly rainfall (Geneva, Freiberg) negative for all months, for the most diverse abdominal and thoracic organs of man (after BOYD) always found negative. On the other hand, in the thermal monthly deviations, the direction of asymmetry in the progress of the months through the year reverses by law, so that it is positive in the winter months, weaker in the summer months, and fluctuating in the intervening months. In the rye ears *u*this upper limb positive, weakens on descending to the lower limbs and turns in the lowest to the negative. It is undisputed that the *m of* all these cases could be taken small enough that the constancy or legality would be disturbed or lost, provided that with the smallness of the *m* the unbalanced contingencies gain a growing influence; but the *meters* , which was at command, has been sufficient to prevent it. But if no essential asymmetry had been present, it would not have existed at any size of the *m*gain such a constant or legal preponderance over the coincidences. The multiple occurrence of such cases first led me to consider the essential asymmetry at all a general role in the area of K.-G. attributable; and undoubtedly the cases of this kind would be piling up, if only sufficient investigations with sufficient *m were available* in relation to it.

XIII. Mathematical relations of the combination of essential and nonessential asymmetry.

§ 96. Let any value *H be taken* as the initial value of the deviations, and if there exist asymmetrical W. (essential asymmetry) with respect to it, then without the occurrence of unbalanced coincidences (random asymmetry) the difference *u* between the mutual deviations would be simply proportional to the magnification or Reduction resp. grow or lose weight. In fact he is at a given output - *m* equal to *x,* as he would in *n* same value -maliger repetition of the observation of each new copies of the same object *xn* times reach, and therefore even when the composition of the *n* Observational series turn into a single continuous difference *x* in *nx* . If, on the other hand, the essential asymmetry were completely eliminated, and the difference merely depended on unbalanced contingencies, then if we found the difference y at the starting m , this difference would not be equal to ny for n times mbecause the direction and magnitude of the difference in the repetitions changes at random, and if, generally speaking, an overweight remains indeterminate to which side, this, that is the definitive difference, changes as long as one moves in large numbers of deviations , and on average even with small numbers, according to a known principle instead of in proportion n rather in proportion . We now introduce the comparable to n -fachende m as the unit of comparison n -fachung and designate the size of the n -dependent values of $n$have to set as an index, we will [1)]:

in the case of merely essential asymmetry:

$$u_n = nx_1 \quad (1)$$

in the case of merely insignificant asymmetry:

(2)

and in the case of the meeting of both:

(3)

where $y_{1 \text{ can be}}$ generally spoken with x_1 equal or unequal sign; for while x retains its positive or negative direction in the transition from x_1 to nx_1, $y_{1 \text{ may,}}$ at random, maintain or change direction in y_1 without a general decision in between; and if we take y_1 by absolute value, we shall have to set with regard to this doubtfulness:

(4)

and at the starting m itself, where $n = 1$,

$$u_1 = x_1 \pm y_1 . \quad (5)$$

If we now set $n = 100$, another time = 1: 100, we will get:

$$u_{100} = 100\, x_1 \pm 10\, y_1 , \quad (6)$$

, (7)

So when hundredfold increase of the output m the output, according to (6) x to 100 times, the output y only to the 10-fold increase, and should n be increased indefinitely, so would the definitive $y,$ ie the difference dependent on unbalanced contingencies, vanishing altogether against x dependent on essential asymmetry ; conversely, by (7) reducing the output m to 1 **:**100, the output x is

1 : 100, the output *y is* only 1 : 10, and the former would noticeably disappear altogether on the further reduction of *m* , which only in so far as is not quite parallel with the enlargement of *m* , as *m increases* to infinity, but can only be reduced to 2, should still be Difference *u* exist. But generally it follows that the essential asymmetry lighter at large, the minor with a small *m* outweighs unless we the former as an enlarged strongly conditions this as a reduced in strong relationships starting *m* can consider that one may always take it , of which, of course, the need depends, the largest possible*m in* order to obtain the essential asymmetry as undisturbed as possible of minor importance.

1) Here, the value *x* has the index 1 consistently with the above notation, insofar as it designates the value of *x* which takes place at the outgoing *m* , where n = 1 , corresponding to *y.* [Note also that formula (3) only wants to give a schematic representation of the mixture of essential and negligible asymmetry without stating that y_1 represents the same value as in (2). In fact, both values are different. For the term y_1, which is based on an insignificant asymmetry, is nothing more than the average fluctuation of the value of u_n, which is to be expected according to W. whereas the member nx_1, which is based on the essential asymmetry, is the most probable. Represents value of u_n ; the average expected fluctuation around the most probable value, however, depends on the latter and thus has different values, according to which the most probable value is equal to zero or represents a finite quantity. Comp. on this the addition to the following chapter, (§ 101).]

XIV. Formulas for the mean and probable value of the difference dependent on purely random asymmetry *u* .

§ 97. If there are already features for distinguishing the essential from the insignificant asymmetry, it must be admitted that they have no absolute character. Nor, indeed, can one assure absolutely that there is an essential asymmetry, but only that there is an overwhelming probability of it, and the more so the more distinct the above-mentioned distinguishing marks from the accidental.

In order to make a somewhat more definite judgment of probability, it is useful to know what difference one can expect to find after W. and, on average, even at essential symmetry, according to mere chance.

By probable difference I mean those who in a large, strictly speaking, infinite number of cases fall just as often (not reached) as are exceeded; below average or average, obtained by multiplying the values of u obtained in many repeated experiments with given m without regard to the sign, and dividing by the number n of repetitions made. In fact, if one or other of both values has been generally determined for the case of substantial symmetry, then one will find that each value of u obtained in a given mean determinationcompare with it. If he outweighs those values in strong proportions, it will be very unlikely to find that he could be reached in symmetry, because the improbability of this increases with the magnitude of this transgression, and against this a very significant asymmetry from the sign of u *is* very likely to hold. If it remains considerably below these values, one has to conclude with great emphasis on symmetry or small asymmetry of doubtful sign. Yes, one can draw even more precise conclusions. The theory teaches and the experience confirms that the probability relations, which according to GG exist for the observation errors in the sense of the known integral, can be represented with essential symmetry on the$u$ can be transmitted in such a manner that the exceeding of middle or probable u to equal to W given limits . is subject to the exceeding of the simple mean or probable observation error.

This will be shown in greater detail and more accurately in the two following chapters, proven empirically and the application thereof shown. Here I limit myself to presumably borrow from it the following principal provisions which are suitable for giving the most general indication.

§. 98. One has to distinguish two cases, the only ideal case in which the values Δ are calculated from the true A , as would be obtainable from an infinite number of individual values , that is, in the absolute normal case, and the case of reality, where they are from that in some way incorrect Acan be expected, as it can be obtained from a finite number of values. At first it does not matter which law of distribution the individual values obey, not the size, only the number of the same in the case of the + and - comes into consideration, and one can use the known sack with an equal number of white and black Use balls instead of + and - as a stop for calculation. Last case has for the theoretical calculation of the average and probable *and* a particular law of distribution were assumed, because thereafter the average and reasonably probable error of the false from the true A directed, and this again to the size of the average and probable $u$is of influence. Accordingly, secondly, for the distribution, we subject the GG to random deviations from the observation means represented by the known integral, since this distribution is considered normal for the ideal case of a substantially symmetric K.-G. can apply.

Let U *be* the mean, V the probable u in the sense just given (§ 97) on the assumption of the first case, $\boldsymbol{U}$ and $\boldsymbol{V}$on the assumption of the second case [1] , then one has, to a very small m remarkably correct, the following normal rules:

, (1)

, (2)

, (3)

, (4)

log 0.799788 = 0.90194 - 1, log 0.67449 = 0.828 97 - 1,

log 0.48097 = 0.68212 - 1, log 0.40659 = 0.60916 - 1.

In the values of *U* and ***U*** , the upper sign of 0.5 and 1.5 respectively is to be used odd, the lower one for even *m* .

[1)] *V* and ***V*** therefore have a different meaning here than those specified in § 10.

§ 99. The following remarks. All four formulas are principally derived only as approximations for larger *m* , and in this derivation the ± corrected corrections 0.5 and 1.5 of the values *U* and ***U*** (which disappears against larger *m*) are not found. But it is empirically found that, by affixing them, the formulas in question are much less noticeable down to much smaller *m* - almost to the smallest - than without them.

A result of the correction ± 0.5 for *U* is that its value is the same for every odd and the next largest even *m* , and a success of the correction ± 1.5 for ***U*** , that the value for each odd and that around 3 units larger straight *m is the* same size. By falling to quite precise formulas for *U,* but which become too cumbersome in the case of larger *m* , it can be proved that the first success is normally from the smallest to the largest *m*is strictly and universally valid; As for the second, I can not do the same thing with equal certainty, but only according to those in Ch. The following empirical results claim that show this success as close as one might expect from the uncertainty of such results; and the theoretical derivation of the given formulas for ***U*** and ***V is*** not quite as certain as for *U* and *V,* and since these are the only practical applications for our present investigation, whereas those for *U* and *V*In this respect, the empirical probationary results for ***U*** and ***V*** obtained by me in a very peculiar, very laborious method in § 115 must be referred to in this connection.

It will be useful to note that the previous formulas can also be applied in the case if, instead of *m* a single series the summatorische ∑ *m* more, with respect to various funds received series, either with the same or different *m* has the right to then substituting ∑ *m* for *m* in previous formulas; only the condition must be fulfilled, that the contingencies, which have an influence on the size of the *u* in the individual series , can be regarded as equally independent of each other, and thus in aggregation of the different *m*corresponding to the compensation, as if one increases the *m of the* same series.

§ 100. We still wish to raise some theoretical reservations that could easily be imposed by looking at the previous formulas.

After the assumed in previous formulas equal probability of Δ 'and Δ , had been in the sack with endless white and black balls, which us the Δ ' and Δ , can represent to accept an equal number of both; and if the whole infinite number were drawn, that is, the *m of* the train would be infinite, then the difference *u should be* zero and indeed zero for every repetition of such a move, so also the mean and probable difference be zero, whereas the formulas one , with *m* growing indefinitely and at $m = \infty$let infinite value of *U* , *V,* ***U*** , ***V*** find.

From other side, but one that with growing lights, *meters* and the scope of a possible accidental difference between μ' and $\mu_,$ increases, and in this respect, however, growth of the middle and probable difference with *m* can be expected, which is anticipated no limit, hereafter In the infinite *m,* an infinite difference can indeed be expected.

This apparent paradox stands out in that, though the average and probable difference at infinite *m* the formulas to itself becomes infinite according he but as with proportional, as the size of second order compared *m* both μ' and $\mu_,$ which even with *m* are of the same order, disappears, so that from this mathematical aspects the widest possible μ', which can be drawn, still equal to $\mu_,$ or $\mu' : \mu_,$ can set the unit the same as a condition of symmetry hold, if already μ' of $\mu_,$ differs by one against both vanishing size.

Also, one may perhaps conceive the matter as follows: Since an infinity multiplied by an infinity can be thought of, which again gives an infinity, it follows that one simply draws an infinite number of orbs, not that one pulls the integer, and it After all, in absolute infinity the number of white and black spheres could be equal, without this equality occurring at $m = \infty$, unless the ∞ signified absolute infinity.

In any case, experience can not be reconciled with anything other than the above form of formulas, and thus it justifies itself against any reservation of theory, which might be left over from the previous point of view.

Second, one can establish that, as with increasing *m* the difference between the true and false *A* more and more reduced and at infinite *m* which the wrong is vanishingly small but according to the above formulas *A* projected ***U*** to the true *A* calculated *U* one at larger *m* noticeably constant ratio, whose exact limit for infinite *m* instead of 1 rather

(5)

is.

The reason for this, however, is that the number of deviations between the true and the false mean, and on which the difference between *U* and ***U*** depends, decreases with the approximation of the false to the true mean, but with the size of the *m* to; and insofar as the approximation of both means is conditioned by the size of *m* , this is compensated for in such a way that that constant ratio comes out with increasing m ; and even with infinite approximation of both means, by infinity of the m even an infinite set of infinitesimal deviations between the two are thought to lie mathematically. Incidentally, experience is also crucial in this regard. According to the values of U and $\boldsymbol{U}$, which are comparable with each other , we find for m = 10; 50; 100 of the series still the value $\boldsymbol{U} : U$ equal to 0,554; 0.558; 0.608, which deviates from the theoretical ratio and from the constancy only within the limits of the expected uncertainty, which of course is considerably greater for the ratio of two values than for the individual values.

Third, the following circumstance may be noticeable. Depending is expected deviation from true or false means the sum falls different from the same, averaging the smaller in calculation of the wrong remedy for the bill from the true center, the smaller m and the false therefore is the means. But the difference is almost vanishing even in moderate m , in that, as I have shown theoretically and empirically in a special treatise [2] , the false to the true sum behaves on average as to what proportion with increasing $m$the unit is rapidly approaching. Against this it is noticeable that the mean difference between the number of positive and negative deviations is so considerably different than the above limit ratio $\boldsymbol{U} : U$ =0.6028.

2) ["About Corrections Regarding Accuracy Determination of Observations" etc. in the reports of Kgl. Sächs. Society of Sciences. 1861.]

This can be understood as follows. If the deviations actually obtained could be calculated by the true mean, then at finite m not only the number, but also the sum of them on both sides, would be randomly unequal. Now the determination of the wrong mean happens in such a way that the sums of the Δ It is artificially similar on both sides, since this is the condition of the arithmetic mean, and one would have to expect that with the difference in sums the difference in numbers would disappear altogether if the wrong means were taken into account, if both differences were proportional. This is not the case now; but in any case one sees that the disappearance of the total difference in the transition from the true very well may be associated with such a significant reduction in the number of difference at the wrong means as in the relations $\boldsymbol{U}$: U turns.

As far as the essential asymmetry is concerned, it only takes a small share in this reduction. As noted above (Chap. XIII), neither substantial nor negligible asymmetry can develop at too low *a level* ; but as the deviation of the false from the true average is on average as much in the sense as in the sense of the essential asymmetry, a large m compensates the influence of this for the essential asymmetry.

§101. [Additive. Finally, in order to give the modifications which the above formulas suffer in the case of substantial asymmetry, and at the same time to prove the validity of the scheme given in the previous chapter of the mixture of essential and negligible asymmetry, it should be noted that in the case of substantially asymmetrical K.- G. not based on the arithmetic mean, but on the densest values in principle. With respect to the latter value, the chances of positive and negative deviations are then not the same, but, in agreement with the theoretical determination of the densest value in conditions of mutual simple average deviations e' and $e_,$ to accept. Because the proportion $e' : e_, = m' : m_,$ defines the densest value, so that the total number of copies is distributed in the ratio $e' : e_,$ on both sides of the most dense value, and hence this ratio is the probabilities p and $q = 1\ p$ for positive and negative deviations determined. It is accordingly for a K.-G. with given e' and $e_,$ bez. the densest value [3] :

; (6)

Then, first, the most likely difference between positive and negative deviations for any *m is the* same:

$$m\ (p - q).\ (7)$$

Further, if the mean and probable deviation from this value are similarly denoted by U and V , as was the case above for the mean and probable deviation from the zero value, the corrections are obtained, with the following omissions:

(8th)

$$V = 0.6745 \cdot \quad (9)$$

Thus, the probable limits of the differences *u are* equal

$$(p - q)\ m \pm 0.6745 \cdot \ ,\ (10)$$

ie, bet 1 against 1 that an observed *u is* greater than $(p - q)\ m - 0.6745$ and less than $(p - q)\ m + 0.6745$.]

[3] [A more detailed discussion teaches that in weak asymmetry the one arithmetic treatment of the K.-G. allowed, p and q only order quantities of order 1: where *m is* the total number of copies of K.-G. is different from ½.]

[This determination of the probable limits reveals at the same time the relations of the essential and insubstantial asymmetry, if, in agreement with the statements of the preceding chapter, the most probable difference between values of *u and* insignificant asymmetry is understood to mean the probable variation around this most probable value , It shows that in formula (3) of the given chapter $x_1 = (p - q)\ m;\ y_1 =$

0.6745 , and then, in formula (2), where $p = q = ½$, $y_1 = 0.6745$ has to put.]

[You reach the specified provisions of likely *u,* and the average and probable fluctuations around this value, considering the likelihood that under *m* deviations m' positive and $m_,$ negative found that thus $u = m' - m_,$, equal:

(11)

and assuming a large value of *m* sets the approximate value :

(12)

derived]

XV. Probability determinations for the purely random asymmetry dependent difference *u* at the output of the true mean.

§ 102. In general, K.-G. between the number of positive and negative deviations μ', $\mu_,$ bez. of the arithmetic mean *A* a difference $u = \mu' - \mu_,$ of which one wonders whether it is not explainable by unbalanced coincidences due to finiteness of *m* in the case of essentially the same deviation of the mutual deviations , or if the participation of an asymmetric law of the Deviations on both sides are to be considered as contributing, since unbalanced contingencies in the finite *m,* with which one has always to do, can not be missing at all, without that however they need to condition the found difference alone. This can be used to specify probability determinations which, for the reason given in § 94, have no fundamental importance for our doctrine, but nevertheless an interest which causes me, without exhausting this subject here and to pursue it in its mathematical depth, to a certain extent to respond to it.

The most general thing that can be said about it is that the greater the difference *u* in terms of the absolute value is in relation to the total number *m* , and the larger *m* itself, the less likely it is to become dependent on mere unbalanced contingencies, or, as we have said briefly the mere chance of difference, the more likely the co-dependency of asymmetrical W., without, of course, being able to attain an absolute certainty in this way. But probably can specify up in much symmetrical W. the random medium and probable difference how big *u*between μ' and $\mu_,$ is that according to the existing*m* can be expected, if by mean differences, *U* , is meant the difference, which, with repeated repetition of observation under the same circumstances, with the same *m* from ever new specimens of the same object, as the arithmetic mean of the various values of *u* (the absolute values after); among probable differences, in short *V*, the value which is exceeded or *fallen* below as often as it is, of which the first with respect to the *u* values is the same as *A* ref. the *a* value, the second the same as the central value bez. the *a* Values is. In ever stronger circumstances now, according to the theory of probability, determinable, purely accidental mean and probable *u* in a given distribution table, resp. *U* and *V,* beyond which *u are* found , become less likely to be dependent on mere chance; and even after the ratio of this excess, degrees of improbability can be given, for which the rules are known to mathematicians, to which I will not go into detail here.

Now it seems at first, of course, in determining the ratios of *u* assumed known from the urn of probability on the condition that it infinitely many in number but the same number of white and black balls contained by at drawing, each *m* a balls equal W . for the train of white and black balls, according to which the number difference *u of* the balls would have to be zero, but at random, say *n* moves of every *m* balls, soon the number one, soon the other balls soon more, now less predominates, short a random difference *u* of random size in a random direction. It can not only be calculated, but also proven by experience, how great in the case of many (strictly speaking, infinitely many) trains are the mean and probable *and* the absolute values, and it stands to reason, the middle and probable Value of the *u* , which, by mere coincidence, between the number of positive and negative deviations from the arithmetic mean of a K.-G. assuming symmetrical W. with respect to the same. Now, however, a circumstance will continue to be given (§ 109), which makes the mere transference of the result from one case to another impossible; but let us proceed from the case just discussed, in which some interesting, if not mistaken, circumstances will be found out of the past, only to pass later on to the more complicated ones, which the collective deviations present; Let us first briefly describe the result of the procession of the spheres from the urn, under the given conditions, and in regard to the results for larger *m* based on phrases which I find in POISSON's "Recherches sur la probabilité des jugements" and HAUBER's treatises in the 7th, 8th and 9th volumes of the BAUMGARTNER and ETTINGSHAUSEN journal for physics and mathematics, and which are also undoubtedly elsewhere [1)] , whereas for smaller *m,* for which, to my knowledge, there is no investigation , I am based on my own investigation.

[1)] [Eg in MEYER's lectures on probability calculus, in connection with the treatment of BEBNOULLI's theorem; Cape. III.]

§ 103. First, I find in these sources the general result that the probability relations of *u* for very large *m* and *n* under the conditions given follow in their relations the same law of random deviations as the deviations Δ from the arithmetic mean to the GG the observation error, and that if Q 2 - *is* the mean of the squares of all possible *u* given *m* , then also between *Q* , *U* and *V* at large *m* and *n*the same ratio exists as GG between *q* 2 *, ε* and *w,* if *q* 2 is the mean σθυαρε error $\sum\Delta^2 : m$, *ε ισ* the simple average error $\sum\Delta : m$, and *w is* the probable error. What:

$U =$ $= 0.79788\ Q \log 0.79788 = 0.90194 - 1$ (1)

$V = 0.677449$ Q $\log 0.67449 = 0.82897 - 1$ (2)

$V = 0.84535\ U \log 0.84535 = 0.92703 - 1$ (3)

After its own investigation but I find the following two, not uninteresting in itself sets which for very large, strictly speaking, infinite *n* remain strictly valid, like *m* be large or small, will therefore find themselves approximate the more the more often the train one of each *m* balls repeated, be it 2 or 10 or 100, etc. each time:

1) that $Q^2 = m$

2) that *U* is equal for a given odd and even greater by 1 greater *m,* so for *m* = 1 and 2, 3 and 4, 99 and 100 usf.

§ 104. Following is the way how to come to previous sentences mathematically.

Each time *m,* for example, 4 balls drawn from the urn in question, the following 5 cases can occur:

Special number of drawn white and black balls		*u*
4 w.	o black	+ 4
3 w.	1 black	+ 2
2 w.	2 black	0
1 w.	3 black	- 2
0 w.	4 black	- 4

In general, for given *m,* are the potential *u* values *m* + 1, when the positive and negative *u* are distinguished, however, only ½ *m* + 1 for an even *m,* ½ (*m* + 1) for odd *m* when the *u* for absolute Values, ie positive and negative, are counted as equal. For any not too large *m* , the possible *u are* easy to find empirically according to the previous scheme, and it is now asked how often with very frequent trains of *m,* that is, of 4 bullets, each of the possible *u*in proportion to the total number of possible *u* , or in short, which W. has each *u* . Set it to W . found in the same way. If one then multiplies each *u* by its W. and adds these products, then according to the known principle of probability calculus, one has therein the exact mean *u* , which we call *U.* At first it may seem that the sum of those products even with the sum of W . should be divided to the middle *u*to obtain; but every single W. presents itself as a fractional value of 1, and the total sum of these fractions gives 1, which does not require a special division. Similarly, one obtains the mean u^2, which we call Q^2, by summing the products of the individual u^2 in their respective W.

So, to find U and Q^2 for a given m, take the possible u in the above example, determine the W. of each as follows, and then take the sum of the products as indicated.

In order that a W. u, short $W[u]$ or $W[\mu' - \mu_,]$, under separation of positive and negative values for given m gain, you have the following, known to mathematicians formula[2]:

, (4)

where 1 . 2 . 3 ... m *is* the product of all integers from 1, to incl. M, corresponding to μ' and $\mu_,$ but in the case that μ' or $\mu_, = 0$, the value is 1.2.3. . μ' or 1.2.3 ... $\mu_,$ is to be set equal to 1.

2) Shorter one expresses the same formula as follows:

Applying this to our example $m = 4$, take μ' for the number of white, $\mu_,$ for the black sphere, $1 \cdot 2 \cdot 3 \cdot 4 = 24$; ; so we get:

μ'	$\mu_,$	u	$W[u]$
4	0	+ 4	
3	1	+ 2	
2	2	0	
1	3	- 2	
0	4	- 4	

Now we take u to absolute values ruthlessly on its sign, as we have to do, because U is taken as the average of the absolute values, doubled, for odd m the W. for each, *and,* for an even $m,$ as at $m = 4$, for each u with. Exception of $u = 0$, and we have to write the previous example like this:

$\pm u$	$W[\pm u]$
4	

2	
0	

The corresponding implementation for the odd m = 5 and 1 larger straight m = 6 gives:

for m = 5

$\pm u$	$W[\pm u]$
5	
3	
1	

for m = 6

$\pm u$	$W[\pm u]$
6	
4	
2	
0	

[This implies that $U = 1\frac{1}{2}$, $Q^2 = 4$ for $m = 4$; $U = 1\,{}^7/_8$, $Q^2 = 5$ for $m = 5$, and $U = 1\,{}^7/_8$, $Q^2 = 6$ for $m = 6$, so that there are confirmed the above rates by $Q^2 = m$ for $m = 4$, 5 and 6, and U receives the same value for m = 5 and 6. In the same way, for any other $m$confirmation by direct invoice.]

[But to prove the two theorems in their general validity, denote Q and U *in* terms of the dependence of m on Q_m and U_m, and put first:

, (5)

where the summation over all pairs of values $(\mu', \mu,) =$ $(m, 0)$; $(m-1,1)$; ···· $(1, m-1)$; $(0, m)$, for which $\mu' + \mu, = m$. Thus, $(\mu' - \mu,)^2 = (\mu' + \mu,)^2 - 4\mu'\mu, = m^2 - 4\mu'\mu,$, and one obtains by substitution of the latter value:

, (6)

There

.

if $\mu' = 0$ or $\mu_, = 0$, then the second sum is only over the value pairs $(\mu', \mu_,) =$ $(m - 1, 1)$, $(m - 2, 2)$, $\cdots$ $(1, m - 1)$, and one can therefore represent Q_m^2 in the following form:

, (7)

But the first sum is equal to $(1 + 1)^m : 2^m$, the second equal to $(1 + 1)^{m-2} : 2^{m-2}$, as immediately recognizable when the dividends are developed according to the binomial theorem, and the value of each of the two sums is equal to one. Therefore you get:

$$1) \; Q_m^2 = m^2 - m(m - 1) = m .$$

Assume further that for a straight m, which is assumed equal to 2μ:

(8th)

for the smaller odd $m = 2\mu - 1$ by 1:

(9)

and at first extend the summation over the pairs of values: $(\mu', \mu_,) = (2\mu, 0)$, $(2\mu - 1, 1)$, $\cdots\cdots$ $(\mu + 1, \mu - 1)$; secondarily via the pairs of values $(\mu', \mu_,) = (2\mu - 1, 0)$, $(2\mu - 2, 1)$, $\cdots\cdots$ $(\mu, \mu - 1)$. In the former case $\mu' = \mu + 1 + 1$, $\mu_, = \mu - 1 - \lambda$, in the latter case, set $\mu' = \mu + 1$, $\mu_, = \mu - 1 - 1$, where, in both cases, λ must assume the μ values $\mu - 1, \mu - 2, \cdots 0$, so that the following forms of representation are given wins:

; (10)

; (11)

But for any positive integers μ and v 3) :

, (12)

so is also:

(13)

and you get by simple reduction:

1. .]

§ 105. In the previous two sets nothing is contained on the speed relationship which, in the formulas (1), (2) (3) due to the applicability of the GG to the likelihood ratios *and* between the values of *U, Q* and *V* set and, as yet, there is no [simple] dependence of the values *U* and *V* on the size of the *m* , as we need it. But if we substitute the value for *Q* in the above formulas on the basis of Theorem 1) , *we* obtain the following two formulas which do what is required [4)]:

$$U = 0.79788 \quad (14)$$

$$V = 0.67449 \quad , (15)$$

By the way, formulas which can be derived from general formulas of the displayed sources, so that nothing essentially new is offered; against this can be set to 2 *)* establish the following, I think, previously unknown correction of the formula (14), to which the following to premise.

3) [It proves this identity by first

sets and then in turn

for $\lambda = 1, 2, \ldots \mu - 1$

replaced.]

4) [The same formula for *U is obtained* when, in the above representation of $U_{2\mu}$, for the sake of simplicity, in the unreduced form

assuming, according to the STIRLING formula $(2\mu)! = (2\mu)^{2\mu} \cdot \exp[-2\mu]$ and $\mu! = \mu^{\mu} \cdot \exp[-\mu]$ sets; then the required reduction is obtained

or .

However, since only an approximation of the true value of $U_{2\mu} = U_{2\mu - 1}$ is achieved, it is appropriate for smaller values of 2μ or $2\mu - 1$, based on the more precise formula

.

the approximate values of $(2\mu)!$ and $(\mu)!$ still the factor

respectively.

to add; then you get

;

thus for straight m the formula:

;

for odd m the formula:

,

Thus to win in this way are listed below (16) correction for U.] .

While the above sentences 1) and 2) remain valid for arbitrarily small and large m with only sufficiently large n , the formulas (14) and (15), as well as the formulas (1), (2) and (3), out of which they follow a great, strictly speaking, infinite m , without demanding a greater n than 1. But they wanted to them on such a small m apply as 3, 4 or 5, they would even in the middle of infinitely many trains, so in an infinitely large n a remarkably bad result, however, already in a unique part of a very large m a noticeably correct result give. But if we replace the formula (14) with the following:

$$U = 0.799788 \qquad (16)$$

using the upper sign for the straight line, the lower one for odd *m,* we thus meet the requirement of Theorem 2, and at the same time find empirically that this formula, even down to the smallest *m* , is not absolute, but almost exactly the exact theoretical Numbers are correct, which are obtained in the above-mentioned way in principle exactly the same for small as for large *m* , only that for large *m,* the bill is no longer feasible. In fact, the following comparison table is given below:

Comparison of the exact values of *U* with those calculated according to (16).

m	exactly	0,797 88	diff.
1 u. 2	1.0000	.9772	- 0.0228
3 u. 4	1.5000	1.4927	- 0,0073
5 and 6	1.8750	1.8712	- 0.0038
7 and 8th	2.1875	2.1851	- 0.0024
9 u. 10	2.4609	2.4592	- 0,0017
11 u.12	2.7070	2.7058	- 0,0012
15 u.16	3.1421	3.1413	- 0.0008
25 and 26	4.0295	4.0291	- 0.0004

As you can see, all the values of *U* in minus calculated from formula (16) deviate from the exact ones, but even at *m* = 1 and 2 the deviation is very insignificant, at *m* = 25 and 26 there are only 4 units of the 4. Decimal and decreases as the *m* increases. Of course, the uncorrected formula (14) gives much larger deviations from the exact value for small *m* ; at *m* =25 it is still - 0.0401, at *m* = 26 still + 0.0389; and only at much larger *m* does it become noticeably vanishing according to formula (14) as in formula (16).

§ 106. As far as the value *V* is concerned, the same would be given in principle precisely by determining the value *u* , with respect to which the probability of greater *u* equal to the probability of smaller *u;* but if we try to apply this to examples of small *m,* such as the above with *m* = 4, 5 or 6, then they do not produce such a value, but what values we want to take for them, then the probability sum of the larger and smaller *u is* unequal, and if one had the same, if one requires a certain value at all, between two of the *u* to search, which are each separated by 2, z. For example, at *m* = 5 between *u* = 3 and 1, at *m* = 6 between *u* = 2 and 0, without, as far as I can see, a rational principle for a more exact determination exists, which does not prevent, with such a large *m* that ± 2 disappears, however, to find the formula (15) permissible. Meanwhile, seemed of interest, a provision for smaller *meters* to try the following principle.

The number of values *z,* which have a value *a of* a K.-G. is written, be it in a primary or reduced panel, is to think of earlier disputes actually distributed over a whole interval whose boundaries fall in equidistant *a* in the middle between two *a* . If we now compare the equidistants *u* with the equidistants *a* , we can by analogy think of the probabilities which are given to *u as* being distributed over an interval of size 2, and hence in the same way as we do of the central value of *a*by interpolating the interval into which it falls (see § 82), we find the central value of *u,* di *V;* by interpolating its interval. I do not say that this consideration is strict; because that distribution of *z* at K.-G. is given as necessary by the nature of the thing, but in the case of the *u* in itself it is not required by anything, and a determinate finding by interpolation should not be confused with an exact one. In the meantime, however, the attempt was made to find out what came of it, and the values thus found for given *m could be combined* with those for large *m*given by formula (15). Instead of merely interpolating with first differences, I have applied the more precise with second differences and obtained the following results:

Comparison of the interpolated *V* with those calculated according to (15).

m	interpolates	0.67449	diff.
2	1.0000	.9539	- 0.0461
3	1.1716	1.1682	- 0.0034
4	1.3837	1.3490	- 0.0347
5	1.5072	1.5082	+ 0.0010
6	1.6667	1.6522	- 0.0145
7	1.7912	1.7845	- 0.0067
8th	1.9117	1.9077	- 0.0040
9	2.0372	2.0235	- 0.0137
10	2.1328	2.1329	+ 0.0001
15	2.6168	2.6123	- 0.0045
20	3.0241	3.0164	- 0.0077
25	3.3733	3.3724	- 0.0009

It can be seen that the comparison is indeed not unsuccessful, in that the *V* values obtained by interpolation agree almost exactly, even at very low values of *m,* with those which correspond to formula (15). And it remains only conspicuous that the differences between the related values do not follow a regular course, and, while most of the values calculated by (15) are smaller by a trifle than the interpolated values, with a few (for m = 5 and 10) the reverse takes place, which is not due to oversight, as I have convinced myself by careful revision.

[However, it is precisely this continuous agreement that shows that the interpolation determination is true only in so far as formula (15) represents the probable value of u with sufficient approximation. But since this - following the derivation of that formula - is only the case, if quantities of order 1 : may be neglected, then for smaller m neither formula (15) nor the method of interpolation will be used with advantage, but rather rather to more precise provisions of Vhold each other. Such can be obtained in successive approximation to the true value by means of the molecular formula of MAC LAURIN, which is also called EULER's empirical formula. The fundamental meaning of this empirical formula is that it reduces the calculation of a discrete sum, upon satisfaction of certain conditions, to integration and differentiation, and thereby substitutes a constant change for the cumulative value that changes step by step from interval to interval. If this is done for the sum of the values W [± u], then the u can be determined up to which the sum of the values above and below is equal to ½, whereby V *is* found.]

[It now follows, as stated in the first supplement (§ 110), for even and odd m:

V = 0.674 489 - 1; (17)

if quantities of order 1 **: are** taken into account, those of order 1 : m *are* neglected. If you take the sizes of the order 1 : m further you will find:

1.for even $m = 2\,\mu$

; (18a)

2.for odd $m = 2\,\mu - 1$

; (1

8b)

where the value of c *by* means of the t- table in both cases for a given μ = ½ m resp. ½ $(m + 1)$ from:

(18c)

can be found. The two formulas (18a), (18b) form the analog to (16); they have the consequence that the V*s* for a straight m and the next successive odds are almost equal and would be completely equal if c , neglecting the term 1 : 16 μ , would be set equal to 0.67449 in (18c).]

[For comparison of the three approximate formulas (15), (17) and (18), whose V *are designated* in sequence as V_1, V_2 and V_3, the following combination is used:

m	V_1	V_2	V_3
4	1,349	0.349	0,565
5	1,508	0,508	0.529
6	1,652	0.652	0.827
9	2,023	1,023	1,043
10	2,133	1,133	1,267
11	2,237	1,237	1.257
20	3,016	2,016	2,111
100	6.745	5,745	5,786
1000	21.329	20.329	20.333

§ 107. Since, apart from the *Vs* to be produced in interpolation, all previous determinations are based on unequivocal arithmetic principles and propositions, an empirical proof of them in itself should not be necessary, but I want to go into such a case, partly because the method of proving itself It may give a peculiar interest by the substitution of the probabilistic urn, partly because its results give a certain indication of the extent to which the exact values of Q and U for given m, which in principle presuppose a determination from infinite n , are still great finite n, as it is empirically available to expect to find again.

It is undeniable that the urn, with infinitely many white and black balls of equal number, gives a very suitable idea by which to explain the preceding sentences, but such an urn can not be made, and even if it is replaced by an urn with a finite urn Number of balls replaced, in which the *m*Ball back after each move, which may well happen, the process would be extremely boring on very many trains and the production of a quite random mixture of balls before each new course be difficult to achieve, in short the actual application of the method always be practically impracticable; I also do not know that any use has ever been made of it. But the equivalent of the urn can be found in the lists of drawn lottery lotteries, of which the even ones are white, the odd ones are black, or when compared with positive and negative deviations from the same, some are positive, the others can be considered negative.

To this end, in the 1950s, I obtained lists of ten Saxon lotteries from 1843 to 1852, each containing between 32,000 and 34,000 numbers, in the 50s, lists in which the winning numbers were drawn according to the random sequence in which they were drawn were standing, as like 28904; 24460; 32305; 16019; 157; 3708; 16 928 etc. Although the number of numbers of each annual lottery remains only a finite number, and the numbers drawn are not put into the Wheel of Fortune, the drawing of earlier numbers does not alter the likelihood ratio of the later ones, as is the case with the

Urn with a finite number of balls would be the case, and it can be regarded as if there were an urn with an infinite number of balls [5] .

[5] The lot numbers in the fortune-rank, as far as I have been able to observe during a visit to the institute, are small pencils, which, seen in greater detail, are little rolls, consisting of tightly rolled up notes and ring-shaped slips containing the numbers are. Maybe this description after the memory is not exactly what it does not matter here. Before the draw, these numbers are arranged on boards according to their order, 1000 each on a board. These boards are emptied in irregular order, determined by accidental call of an official first in a box and from here in the wheel of fortune, so that from the outset an irregular mixture of thousands instead of has, then the wheel reversed, and this after every 100 drawn numbers repeated. On the axis of the wheel four open-ended wings are mounted, which rotate in the opposite direction of the wheel and thereby convey the irregular crowd. Looking at how this happens and the lots falling apart, one feels tempted to believe that just a few twists are enough to make the mix quite irregular; but, according to the officials, in the first drawings in which the lottery is divided, more and more neighbor numbers are to appear one after the other, whereas in the last drawing, after the quantity has been brought about by a hundred-fold turn of the wheel, nothing of the kind is noticed. as this happens, and the lots fall apart, one feels tempted to believe that quite a few twists are enough to make the mix quite irregular; but, according to the officials, in the first drawings in which the lottery is divided, more and more neighbor numbers are to appear one after the other, whereas in the last drawing, after the quantity has been brought about by a hundred-fold turn of the wheel, nothing of the kind is noticed. as this happens, and the lots fall apart, one feels tempted to believe that quite a few twists are enough to make the mix quite irregular; but, according to the officials, in the first drawings in which the lottery is divided, more and more neighbor numbers are to appear one after the other, whereas in the last drawing, after the quantity has been brought about by a hundred-fold turn of the wheel, nothing of the kind is noticed.

Let us first explain the application of this to the simple case of $m = 3$, where only the two $\pm u = 1$ and 3 are possible with the theoretical $W[u] = 0.75$ and 0.25, respectively, which can be found according to given rules , Repeating the determination of $m = 3$ from new numbers 2000 times , ie $n = 2000$, repeated the following results:

Empirical number, how often a $\pm u$ in n series of $m = 3$ values occurred, compared to the theoretical number

$m = 3$; $n = 2000$.

$\pm u$	theoretically	Empirically
1	1500	1494
3	500	506

If the numbers obtained are divided by *n, the* following provisions are obtained from the previous table:

$W [\pm u]$

$\pm u$	theoretically	Empirically
1	0,750	0.747
3	0,250	0.253

from which Q^2, U, V can be determined , as stated earlier; So z. Theoretically Q 2 = 1 · 0.750 + 9 · 0.250 = 3; and U = 1 ξ 0.750 + 3 ξ 0.250 = 1.5. Accordingly, the following results are to be understood and treated with larger m and different, but always very large n .

Empirical number of how often a ± *u* occurred in *n* series of *m* values, compared to the theoretical number.

$\pm u$	$m = 10$; $n = 5000$		$m = 50$; $n = 1000$		$m = 100$; $n = 600$	
	theoretically	Empirically	theoretically	Empirically	theoretically	empirically
0	1230	1201	112	110	48	46
2	2051	2027	216	217	93.5	104
4	1172	1225	192	194	88	85
6	439	442	158	154	80	67
8th	98	97	119.5	120	69.5	68
10	10	8th	84	65	58	63
12	-	-	54	62	47	51
14	-	-	32	41	36	31
16	-	-	17	21	27	34
18	-	-	9	10	19	13
20	-	-	4	3	13	14
22	-	-	2	2	8.5	8th
24	-	-	0.5	1	5.5	7
26	-	-	-	-	3	4

28	-	-	-	-	2	2
30	-	-	-	-	1	1
32	-	-	-	-	0.5	0
34	-	-	-	-	0.3	1
36	-	-	-	-	0.1	1
38	-	-	-	-	0.1	0
	5000	5000	1000	1000	600	600

The possible values *u* in the previous table are not fully completed for *m* = 50 and 100, but the missing ones , but of noticeably vanishing W., so that a tremendous *n* would have been necessary should such occur one or the other time.

From the previous table, the following table of empirical Q^2 *, U, V is derived* in comparison with the theoretical values.

m	*n*	Q^2		U		V	
		theoretically	empirically	theoretically	empirically	0.674 49	empirically interpol.
3	2000	3.00	3.02	1.50	1.51	1.17	1.18
10	5000	10.00	10.13	2.46	2.49	2.13	2.19
50	1000	50,00	52.02	5.61	5.71	4.77	4.76
100	600	100.00	101.68	7.96	8.05	6.74	6.94

The close agreement of the empirical values with the theoretical ones is undoubtedly satisfactory and only conspicuous that for all values of *m* the empirical Q^2 and *U is* a little larger than the theoretical one, which is probably only the case because the series for the Larger *m were* largely obtained by combining the series obtained for the smaller *m* , so that they could extend their influence on the former, which had to be more significant because of the squaring of the *u* in determining Q^2 than at *U* , where the corresponding is shown to a lesser degree.

§ 108. The foregoing considerations and formulas can often be of useful use in statistical investigations. For example, it is necessary to examine whether the difference between the number of births or deaths or suicides in two different seasons, or between the number of male and female births, or between the number of thunderstorms at two different locations, is purely coincidental or whether the nature of the seasons, the sex, the locality has a significant influence on the size and direction of the difference. Summing up, for both discriminated conditions, a very large number, say m , cases have been observed and found that on one side μ', on the other $\mu,$ Cases are, therefore, the absolute difference u is, it will depend on whether the difference found *and* the absolute values of the probable V exceeds or falls rises, and in what circumstances this is the case, to make probability conclusions following manner.

If the W. of μ' and $\mu,$ equal, and thus the difference *u found, were* purely coincidental, it would be just as likely that it would exceed and support the probable difference Vdetermined for this presupposition of symmetrical W. according to previous formulas , and If the observation were repeated very often with the same m , then on average it would be found to be equal to V ; On the other hand, a purely accidental difference becomes, of course, all the more unlikely, the more it becomes the probable V determined on the presupposition of mere chance exceeds; from this, the W., that he is not merely accidental, the greater, the greater this increase takes place; and if the ratios coincide purely coincidentally with u at large m with the ratios of the observation errors according to GG, also according to a table of the GG, the probability ratios will give the error as a function of the ratio in which the probable error w of them will be exceeded or suppressed , substituting V for w , allows even more definite probability calculations to be made in previous relationships.

I do not think it would be possible to raise any objectionable objection to these general statements; in regard to the particular interpretation but I folgends the ratios $u: V$ give the benefit of its practical utilization, should be part of a perfectly familiar with the theory of probability skilled mathematician probably still desirable in the great ease of incorrect terms and fallacies in this field the fundamental revision.

For example, consider m = 1000 thunderstorms over the same period of time at two locations, for both taken together, observed, on one μ' = 530, on the other $\mu,$ = 470, u = 60; so, according to formula (15), the probable difference V, which we expect by mere chance and, under the same presupposition, of symmetrical W. for u and Δ , can substitute for the *w of* the error-table:

$$V = 0.6745 \qquad = 21.33.$$

This value, 21.33, is considerably exceeded by the differences u = 60 found; 60 = 2.81 V , so it is much more probable than the opposite, that the difference is not purely accidental, but a local influence has a share in its formation, but without being able to find it predominantly likely that he merely It is based on the local influence, but only that there is a local influence of definite direction, which leads beyond the one expected only by chance in symmetrical W. On the other hand, if the difference found, u less than the probable V, z. B. μ' = 505, $\mu,$ = 495, hence u = 10 = 0.47 V, whereas V =21,33, a predominant W would not insist that there be only a fortuitous difference, but that the accidental influence is large enough to outweigh any local influence, whereas there is no probability that the difference found will be it is merely accidental or merely dependent on local influences. In short, it is about the W., whether one or the other influence outweighs, not whether only one or the other exist. But if the W., that the local outweighs, is very great, then, of course, at the same time, the W. is very great, that such a one exists; and thereby become bills of this sort of use for the proof of probability of the existence of other than merely accidental influences. If, on the other hand, the W. predominates, that the random influence outweighs the non-accidental,

If we accept this point of view, and thus refer back to the previous examples, we find, at first, where the found difference $u = 60$ and $V = 21.33$, hence $u: V = 2.81$, according to the table of the GG that the W., the difference u will remain below this value as purely coincidental, and for W. the contrary, as 0.942 against 0.058; and provided that value u is nevertheless reached, you will be able to bet against 6 in round numbers 94, he was not merely accidental. In the second case, where $u = 10 = 0.47\ V$, it is found, according to the relevant table, that the W., the difference, *and the like*will remain below this value as a chance, behaving in the opposite way as 0.249 to 0.751, but unless he has remained below that value, the opposite W. will take place for him to have reached this value as a chance, and will be rounded up only 1 to 3 can bet that a local influence outdid the random, 3 against 1 but the opposite, without being able to bet that a local influence was not present at all. At least I would not know how these conditions could otherwise be handled in a practical and rational way.

Let *W ω βε* the W, that Δ or u , assuming symmetrical W , will remain below a given fraction or multiplum of w or V , then, to give a small excerpt from Table [6,] hereof, of the GG, one has to belong to each other:

u	$W\ \omega$		u	W ω
0,10 v	0.05378		2.25 V	0.87088
0.25 V	0.13391		2,50 V	0.90825
0,50 V	0.26407		2.75 V	0.93638
0.75 V	0.38705		3.00 V	0.95698
1,00 V	0.50000		3.25 v	0.97163
1.25 V	0.60083		3,50 V	0.98176
1.50 V	0.68833		4.00 V	0.99302
1.75 V	0.76214		4,50 V	0.99760
2,00 V	0.82266		5.00 V	0.99926

However, it is necessary to guard against misapplication of the same in the following sense when applying the preceding provision. If it has been investigated, whether any two months or any two seasons, without the remainder, in regard to the number of thunderstorms, nothing will prevent the previous determination as to whether the difference of the two months or seasons one other than mere accidental influence on the number of thunderstorms, should be applied just as if it were the local influence of the location. But supposing that the observation of the number of thunderstorms with given *m has been made* for every 12 months, then, even if the storm is the same for all months, the *u*If two of these are compared at random, they will be different, and two months will be found *among* them, which give the greatest amount, which might easily be so great as to conclude, according to his relation to *V,*to a predominant influence. But this conclusion would be erroneous in so far as, under a larger number of cases, large differences in the differences may occur even in the case of low W. In any case, then the months in question remain suspect because of a specific influence; but to ensure that, in my opinion, the observation would have to be particularly extended to them and z. Up to double the number to see if the probability closure is confirmed [7)] .

[6)] [This table can be found in the Berlin astronomer. Yearbook for 1834, p. 309 flgd.]

[7)] [Comp. to this paragraph the second addition (§ 111).]

§ 109. At the outset, it now appears that from previous considerations and formulas also apply directly to the task, from the magnitude of the difference *u,* which is between the number of positive and negative deviations + Δ and - Δ ref. of the arithmetic mean *A* is to conclude, according to W., whether the difference can depend only on accidental events, or whether an influence is founded in the nature of the object and its conditions of existence, which is superior to the preponderance of the number of deviations, if not already but is partly to blame, or in short, whether significant asymmetry in the difference share. And indeed, if we were assured from the beginning that the deviations of the specimens*a show* from their arithmetic mean *A the* same symmetrical value on both sides, as the white and black spheres at the drawing of the same, the foregoing considerations and formulas would be entirely applicable; but that is not the case after the following considerations.

Let us call, in the sense of a well-known usage, true means A_∞ the means of an infinite number of instances, false mean A_m that is only available to us from a finite number *m.* Let us now set symmetrical W. of the deviations. of the true mean, then both the mutual sums of deviations, as well as the rates of deviation on both sides, will be. It may not be the same at random, and it may not normally be proportional to a change in the total number *m* of deviations, but it may change in a functional relationship in the same direction, that is, increase or decrease [8)] . Now, from a finite number of *a*If the wrong means is drawn, then the difference between the mutual sums of deviation disappears, since that is the essence of the arithmetic mean; In the process, the sums are artificially equalized, and if sums and numbers change proportionally, the difference between the sums on both sides would at the same time make the difference *u*disappear between the two numbers, which is not only not the case in experience, but is also not expected because of non-proportional change. But at any rate, by eliminating the difference between the two sums of deviations, the functionally related difference between the two numbers is reduced to the case where the deviations from the true mean were taken, to which the formulas above apply, and can thus be foreseen and probable value of u bez. of the wrong mean, of which we can only count on it, must be lower for the same m , than for of the true, and that the above formulas can no longer be decisive.

[8] Consider that while the true mean is always to be thought of as being an infinite number of a , yet the number m *of* the differences taken may be more or less finite.

In the meantime, the following two conclusions can be drawn from the above: 1) the W. of a significant influence is, when applying the above formulas, to the deviation difference u . the arithmetic mean of A_m at a given m to accept for even greater than appears according to the above formulas, because V, in proportion to which u is concerned, with respect to A_m in any case smaller than inscribed. A_∞ is what the above formulas apply to.

2) Let bez. the wrong mean A_m as well as bez. of the true A_∞ the presuppositions of symmetrical W apply, but then call the values designated above with respect to the former with u, Q, U, V , if they are rather the latter are determined resp. $\boldsymbol{V}$, $\boldsymbol{Q}$, $\boldsymbol{U}$, $\boldsymbol{V}$, it will only apply, this accordingly as a function of m bez. A_m to be determined as those with respect to A_∞, in order to obtain formulas which can serve for appropriate use.

Section 110. [First Amendment. Determination of the probable difference V using the molecular formula of MAC LAURIN or EULER:]

[This molecular formula is [9]:

, (19)

where $b = a + nh$ and $B_1 = {}^1/_6$; $B_3 = {}^1/_{30} \cdot \cdot \cdot \cdot$ the Bernoulli numbers.]

[To sum up the $W[\pm u]$ according to this formula, it is not the original form (4), but the result of this by the approximation formula:

, (20)

or, if one considers terms of order $1 : n$, based on the corrected formula:

(21)

underlying form.]

[If we first use (20), then for $m = 2\mu$; $\mu' = \mu + \nu$; $\mu_{,} = \mu - \nu$; $u = 2\nu$:

; , (22)

The sum of *W* [*u*] between the limits + *2 n* and - *2 n,* or the sum of *W* [± *u*] between the limits 0 and 2 *n* is thus given by:

, (23)

Now, however, according to (19), if in agreement with the approximation given by (20), members of the order 1 **:** *μ are* neglected:

, (24)

Consequently you get:

, (25)

The right side is given a more convenient form by taking x 2 = μ τ 2 ; n 2 = *μt* 2 ; *dx* = *d τ* substituted. One then obtains as an expression of the probability *W* that:

; or

the determination:

, (26)

[9] [EULER derives it from the Institutiones calculi differentialis, Pars post., Cap.V. - Reproduction. z. B. in SCHLÖMILCH's Compendium of Higher Analysis, second volume, p. 226.]

According to her, the probable value of u, ie *V, is* given by:

, (27)

if *t of* the condition:

(27a)

enough. For it is then W., that $\pm u < 2t$ equals ½. To therefrom t to calculate one set $t = c + \gamma$, and determine c from

.

so that, according to the t- table, it is equal to 0.476 936; then the integral between the bounds 0 and $c + \gamma$ splits into two integrals between the bounds 0 and c and between the bounds $c$and $c + \gamma$, resulting in:

,

But since γ ισ a quantity of order 1 **:** 1 , one obtains a sufficient accuracy if exp $[-\tau^2]$ is kept constant in the extension of the integral and set equal to exp $[-(c+\gamma)^2]$. Thus, after division with exp $[-(c+\gamma)^2]$, it becomes :

or .

Because of this you get [10)]:

, (28)

Since $m = 2\mu$ was initially set, it might seem that this formula applies only to even-numbered m . However, the same result is obtained for $m = 2\mu - 1$, as we can not expect, since only quantities of order 1 **: are** taken into account.]

10) [This formula is also given by MEYER in the lectures on probability calculus in the treatment of BERNOULLI's theorem, p.107.]

[But if one wishes to consider quantities of the order 1 **:** m , then instead of (20) one must use the approximate formula (21) and distinguish the case that m *is* even-numbered from the case where m *is* odd.]

[At the outset, assume (22) that the factor $(1 - 1 : 8\mu)$ is included in the regulations there. One then finds by means of (19) taking the first derivatives:

, (29)

when members of order $1 : \mu$ are left aside. This results, if $n^2 = \mu t^2$, $x^2 = \mu \tau^2$ is set, as an expression of the probability W, that:

or ,

, (30)

To get V from this *we have* to assume $W = ½$, then t from the equation:

(31)

to calculate and

(31a)

to put. Assume $t = c + \gamma$ as above , determine c by dividing equation (31) by $(1 - 1 : 8 \mu)$ or, which is the same, multiplied by $(1 + 1 : 8 \mu)$

, (32)

and find out γ :

, (33)

This equation takes into account that γ a small size of the order of l : is, after division by exp $[- (c + \gamma)^2]$, the simple form:

or (33a)

on which, given that $B_1 = 1 : 6$ and $2\mu = m$, as probable value for even *m:*

(34)

follows.]

[If *m is* odd $= 2\mu - 1$, then if $\mu' = \mu + \nu$; $\mu_, = \mu - \nu - 1$; $u = 2\nu + 1$:

=

, (35)

and the probability that u holds between the boundaries + (2 n - 1) and - (2 n - 1) is determined by:

, (36)

Thus, by (19), if $n = t$, the probability exists :

(37)

that

or . (37a)

If we again determine t from the equation:

, (38)

by calculating c and setting $t = c + \gamma$ as in (32) , the result is:

(39)

with neglect of the members of the order 1 **:** ,

, (39a)

consequently

and finally:

.

hence considering $m = 2\mu - 1$ as a probable value for odd m

(40)

results]

§ 111. [**Second Amendment** . The discussion of § 108 is based on the problem of finding unknown probabilities from a large number of observed cases. The same is related to the reversal of BERNOULLl's theorem, according to which limit values can be given for the unknown W. and at the same time the degree of probability with which the unknown W. can be sought within those limits can be calculated. If one has observed two mutually exclusive events *A* and *B* in a large number *m* of cases and thereby the event *A* μ 'times, the event *B* μ , found times, one can first the W. for the occurrence of the event*A* same $\mu ' : m$, the W. for *B* equal to $\mu ,\ : m$ set without the contingencies that the determination of μ ' and μ , wear adhere bill. In fact, you may $\mu '$ **:** *m* and $\mu ,\ : m$ only as the most probable values of the unknown W. *x* and 1 - *x* If it is probable that when the observations from another series of cases are repeated, the most probable values which now result are in the vicinity of those found earlier. In place of these indefinite constellations, the inversion of BERNOULLI's theorem gives the following provisions.]

[There is the W.:

(41)

that the unknown probability *x* for the occurrence of the event *A* between the limits:

and (41a)

lies; the opposite probability 1 - *x* is then simultaneously between the limits

(41b)

to search; while for the difference *u* to be expected with W. *W* between the mutual number of cases the inequality:

(41c)

applies. In particular, *setting W* = ½ makes *c* = 0.476 936, and the substitution of this value gives the probable bounds for *x* ; 1 - *x* and *u.*]

[Thus, for *m* = 1000 thunderstorms observed in two places during the same period of time, they are probable limits to the values of W., which are expected to cause a thunderstorm in one place or another:

1) in one place 0.541 and 0.519, in the other places 0.459 and 0.481, if at the former place 530, on the latter 470 thunderstorm were observed.

2) at one place 0.516 and 0.494, at the other places 0.484 and 0.506, if the numbers of thunderstorms 505 resp. 495. Accordingly, the probable limits for *u* in the first and second case are 60 ± 21.29 resp. 10 ± 21,33.]

The assumption that the number of observed cases is sufficiently large to permit the assumption that the difference u observed is not purely accidental, but due to the difference of the unknown W x and 1- x, *is* beyond these assumptions is, as already indicated, provided that the most probable values of x , $1 - x$ and u just the observed values μ': m , $\mu,$: m and $\mu' - \mu,$. had]

[But there is no compelling reason to assume that these values are the most likely values. For before the observations were made, every hypothesis about the most probable values of $x$and u possessed the same W., and in view of the observations made, one of these hypotheses may be distinguished from the other only by greater W., but not claim a certainty , It is thus still necessary to determine the degree of W. who possesses the hypothesis that the observed values are the most probable, in comparison to other hypotheses which introduce values other than the most probable ones. This is the purpose of the principle, the ENCKE, in the treatise on the method of least squares [11] in the following form, note that the deviations of observed values from the most likely values are referred to as errors.]

["The two hypotheses which are equally probable and mutually exclusive before the observations made, behave directly like the errors or error systems resulting from them."]

[For the sake of comparison, the hypothesis is that the most probable values of x and $1 - x$ are equal to each other, hence equal to ½, according to which the most probable difference $u = 0$ is to be expected. It then has the actually observed difference u the W .:

, (42)

On the basis of the previous hypothesis that the probable values of x and $1 - x$ resp. μ': $m = p$ and $\mu,$ $m = q$ are, however, results for the observed *and* the maximum value of W., namely:

, (43)

So it behaves the W ,. That the observed u is purely random, ie equality of x and $1 - x$ had revealed himself to the W. that the observed *and* the most likely difference value of the mutual numbers μ' and $\mu,$ representing , as

or how ; (44)

and if you want to bet, the bets must be at the specified ratio.]

11) [Berliner Astron. Yearbook f. 1834, p. 258.]

[On other assumptions the probability determinations are based on § 108. First of all, it should be noted that there *u is* taken into account with its absolute value, and it remains undecided on which side the overwhelming number of cases are to be sought. Then it must be remembered that, assuming that the observed difference *u* is not purely accidental, it is clearly assumed that it will consistently have this value, perhaps assuming greater values (which makes the absence of pure randomness only more probable), under no circumstances Value decrease, in short, the observed value seems to be the lower limit, which is only underrun by pure randomness according to the GG.

If one puts one hand requires the observed difference $u = \pm (\mu' - \mu,)$ is purely random, so there after G . G. W. *W ω* that this value is not reached, and the W. 1 - *W ω* that it is reached or exceeded. On the other hand, supposing that this difference is not accidental, but by its nature equal to *u* or greater than *u,* then W. is that he is reached or exceeded to set equal to 1. Thus, it surpasses W. that the observed value *u* is equal to or greater than *u by* nature *,*the W., he is merely coincidental, around *W ω ,* so that the preponderance probability *W ω* for the absence of pure randomness opposes the probability 1 - *W ω* for the existence of pure randomness, and in this ratio is then betted against and for pure randomness .]

XVI. Probability determinations for the purely random asymmetry-dependent difference *v* at the output from the wrong mean.

§ 112. Let us now proceed to the determination of the probability relations of the random difference which is to be expected between the number of positive and negative deviations from a mean of a finite number of values, if the probability of deviations from the true mean, as it results from a infinite number of values would follow, equal to both sides. Since the false mean, obtained from the finite *m* , deviates from the true mean by a random quantity (in the case of different series soon after one, sometimes to the other), the deviations Δ from both means differ in each series; and it remains true even with calculation of the wrong means the same W. the + Δ and -Δ exist if it were the true mean, but the probability ratios of the difference *v* between the number of them change. This is easily comprehended by the consideration given in § 109, since the false mean is determined by the condition that the sum of the deviations from it is made equal on both sides, whereas in the calculation of the unknown true mean at finite *m* in general is to be assumed unequally. By this artificial adjustment of the sums of + Δ and - Δ , their numbers would also be equalized if number and sum suffered proportional changes, which is not the case; but in any case the difference becomes *v*reduced by the transition from true to false means against the difference *u* .

In order to judge in what proportions this reduction according to W. is to be expected, a definite law must be based on the distribution of the true Δ according to number and size, because the likelihood ratios of the difference between true and false mean depends on this, but the latter again Probability ratios of the difference *v.* Now it is known that for the deviations which the single copies of K.-G. in the case of not too irregular distribution in terms of their mean value, the law of error probability determined by the integral Φ (see Chapter XVII) can be used as the basis for a large *m*and has approximate symmetry, and thus this law will also be used in the following.

§ 113. Up to now, there has been no investigation into these conditions, nor are the preliminary investigations known to me so far to treat the task completely afterwards. In the meantime, we find in the supplement (§ 116) an investigation led by me, according to which the mean square of the difference *v* equal to m (1 - 2: π) is given by $\boldsymbol{Q}^2$, and after the experiential sample to be reported has shown that If this determination itself is very approximate up to a m = 4, one could ask whether from the value of $\boldsymbol{Q}$ the remaining probability ratios of *v*can be deduced correspondingly, as in the calculation of the true mean the probability ratios of *u* from the value Q = . This, too, has been confirmed by experience with sufficient approximation. And indeed, for the probable value of *v,* which is called $\boldsymbol{V}$ [if one uses the interpolation determination for comparison], neither is a correction necessary after this derivation, nor for the value of *V* on derivation of Q ; for the simple average *v* but which $\boldsymbol{U}$ hot, only a slightly larger correction than for the simple mean *u,* which we*U* called. Finally, the distribution table of the individual *v* *is* calculated by number and size approximately enough according to this assumption.

The corresponding fundamental provisions are the following:

$\boldsymbol{Q}$ **2** = m=0.366338m; log 0.366338 = 0.56036 - 1; (1)

$\boldsymbol{Q}$ = = 0.60281 ; log 0.60281 = 0.78018 - 1; (2)

$\boldsymbol{U}$ = = 0.48097 ; log=0.68212 - 1; (3)

$\boldsymbol{V}$ = 0.40659 ; log 0.40659 = 0.60916 - 1. (4)

For the determination of W [± v] one must take the difference of the Φ values, which in the table of the *t* to

and

where Q *is* the above value to substitute; for $W[v = 0]$ but in particular to be $t = 1: Q$ associated Φ value. $W_{\omega}[v]$, ie the W. that the given value of *v is* not reached, is found to be the Φ -value, which for $t = (v - 1) : Q$ and $W_a[v]$, ie the for *v* itself and the values below *v* W., than that to $(v + 1) : Q$ belongs.

In the formula for U , the upper sign of the correction ± 1.5 for odd, the lower for even *m,* and an inference of this correction, as well as the reason of the same is the experiential date, for which, however, the theory still seeks that every value from U for a straight *m* noticeably agrees with the smaller by three units values of U for an odd *m, to* which the documents below follow.

Unfortunately, up to now, control of these approximation formulas with respect to *v is* not as well as with respect to their formulas for small *m,* which are accurate for *u* in the previous chapter ; a more palpable defect than the theoretical reasoning and derivation of the above formulas in the supplement is incomplete, and the correction for U even strange. I would, therefore, offer them with little confidence, if I had not been able, by a very extensive empirical test, to substitute this defect to the extent that one can be sure not to make any mistake in the use of it, although a more detailed justification and revision of the Theory would be very desirable by a mathematician of subject.

Like the previous functional values of *u, the* empirical probation is based on the use of lottery lists, which, however, was more complicated without comparison than for the values of the previous chapter. For it was necessary first of all to translate the numbers of each list into values of + Δ and - Δ in such a way that for the whole list the distribution corresponding to the integral Φ came out, according to number and size at the time of calculation, from the true mean which is given by *t* - Table in appendix §183 is represented; then for any random series of such deviations from given *m*to determine the wrong means to count the positive and negative deviations from this false mean and to take the difference between the number of both as *v* . Somewhat more extensively this is treated in the supplement (§ 117) and the example of a determination of *v* for a randomly taken series with $m = 6$ is given there.

§ 114. Thereafter, first of all, in some tables I let the totality of the empirical data follow, which I received directly concerning our task, in order subsequently to connect the main values derived therefrom, together with the values calculated according to the above formulas. If numerical values often occur with a fractional value of 0.5, this is due to the fact that if, as it happens at times, the wrong means coincided exactly with a true deviation value, the deviation from the wrong mean with +0.5 and -0, 5 had to be counted on both sides , which resulted in a *v* , which fell in the middle between the two distanced values of the *v*- scale, but was then distributed to the two neighboring values with 0.5 each.

I. Number *z*, how often there was a difference *v* between the number of positive and negative deviations from the false mean from *m* values at the *n*- time repetition of the determination.

a) in odd *m*

v	$m = 5$ $n = 2400$	$m = 7$ $n = 1700$	$m = 9$ $n = 1320$	$m = 11$ $n = 820$	$m = 13$ $n = 840$	$n = 15$ [1)] $n = 800$	$m = 17$ $n = 600$	$m = 19$ $n = 600$
1	2,155.5	1,388.5	966.5	552	562.5	?	351	327.5
3	244.5	300.5	324.5	235.5	231.5	?	187	197.5
5	-	11	29	32.5	41.5	?	57	63
7	-	-	-	-	4.5	?	5	10
9	-	-	-	-	-	-	-	2

b) for straight *m*

v	$m = 4$ $n = 3000$	$m = 6$ $n = 2000$	$m = 8$ $n = 1500$	$m = 10$ $n = 1200$	$m = 12$ $n = 1000$	$m = 14$ $n = 850$	$m = 16$ $n = 750$	$m = 18$ $n = 660$	$m = 20$ $n = 600$
0	1950	1040	648	494	379	314	247	179.5	176
2	1050	905	753.5	588	489	382.5	333	325.5	256.5
4	-	55	96.5	112	126	127.5	148	120	130.5
6	-	-	2	6	6	25	20	28	33
8th	-	-	-	-	-	1	2	7	3
10	-	-	-	-	-	-	-	-	1

1) [The values of this column were disfigured by irresolvable contradictions]

II. The same information for some larger values of *m*.

v	$m = 30$	$m = 50$	$m = 100$	$m = 500$

	$n = 400$	$n = 240$	$n = 120$	$n = 24$
0	94	49	19	2
2	169	84	31	2
4	90	51	13	3
6	36	32	22	3
8th	8th	14	18	2
10	3	8th	9	2
12	-	3	5	2
14	-	-	2	5
16	-	-	1	0
24	-	-	-	1
28	-	-	-	1
34	-	-	-	1

The same series with $m = 10$, 50, 100 gave the following results when calculating the deviations from the true mean, which are thus directly comparable to the previous ones, calculated by the wrong mean, while the results given in §107, with reference to other series, therefore with larger *n, are* found.

III. Table comparable to the previous tables for the difference *u* in the calculation of the true mean.

u	$m = 10$ $n = 1200$	$m = 50$ $n = 240$	$m = 100$ $n = 120$
0	301	23	10
2	467	52	17
4	299	44	14
6	102	42	13
8th	29	28	22
10	2	16	16
12	-	17	10

14	-	7	2
16	-	10	5
18	-	0	4
20	-	1	2
22	-	-	4
28	-	-	1

In the two tables for the account of the wrong means is the number *z ′*, how often a *v* the same sign with the departure of wrong had the true means, and the number *eg* , how many times it had the opposite sign, in short, how often a *v* with the wrong *a* was equilateral or scalene, with number *z* = *z* '+ *z* , contracted. Now give the values ***z*** = *z* '- *z* , for the values of *m* = 6 to *m*= 30, because for the others the separation of *z* ′ and z , did not happen. Under ∑ (± ***z***) is a sum of ***z*** on absolute values, under ∑ ***z*** with regard to the sign understood.

IV. Difference *z* = *z* ′- *z* , between the number *z* ′*of* the equilateral mean and the number *z*, *of* the unequal values of *v of* equal size, which join the *z* in previous tables, from *m* = 6 to *m* = 30.

a) in odd *m*

v	*m* = 7	*m* = 9	*m* = 11	*m* = 13	*m* = 15	*m* = 17	*m* = 19
	n = 1700	*n* = 1320	*n* = 820	*n* = 840	*n* = 800	*n* = 600	*n* = 600
1	+ 33.5	+ 0.5	- 33	- 25.5	+ 29	+ 1	- 20.5
3	+ 46.5	- 4,5	+ 9.5	+ 21.5	- 7	- 10	+ 11.5
5	0	+ 1	- 0.5	- 8.5	+ 7.5	- 5	- 15
7	-	-	-	+ 0.5	+ 1.5	+ 3	- 4
9	-	-	-	-	-	-	- 2
∑ (± ***z***)	80	6	43	56	45	19	53
∑ (***z***)	+ 80	- 3	- 24	- 12	+31	- 11	- 30

b) for straight *m*

v	*m* = 6	*m* = 8	*m* = 10	*m* = 12	*m* = 14	*m* =16	*m* = 18	*m* = 20	*m* =30
	n = 2000	*n* = 1500	*n* = 1200	*n* = 1000	*n* = 830	*n* = 750	*n* =660	*n* = 600	*n* = 400
2	- 24	+42.5	+ 20	+ 8	+1.5	- 29	- 35.5	- 16.5	+5

4	+13	+11.5	+ 16	+ 8	+0.5	- 14	- 8th	+ 1.5	0
6	-	0	- 4	0	+3	+ 2	+ 2	- 1	+ 4
8th	-	-	-	-	+1	+ 2	+ 1	- 3	- 2
10	-	-	-	-	-	-	-	- 1	- 1
$\sum(\pm z)$	37	54	40	16	6	47	46.5	23	12
$\sum(z)$	- 11	+54	+32	+16	+ 6	- 39	- 40.5	- 20	+ 6

It may seem a bit conspicuous that the values of z, and thus also $\sum z$, are almost all positive for the smaller, even-numbered *m* values . Probably, however, this has the same reason as was claimed for an analogous phenomenon (§ 107), namely, that the series of smaller *m are included* in the series of larger *m* , so that the series with different *m are* not completely independent of each other, but not only each series for itself, which gave a *v* , but all *n*- series for a given *m* together are arranged purely by coincidence.

§ 115. From the first two tables, the following main values are derived, whose combination with the available theoretical values, according to the above formulas, can serve to test these formulas.

m	Q^2		U		V	
	observed	0.366338 *m*	observed	0.48097	Obs. [2)]	0.40659
4	1.40	1.45	0.70	0.76	0.72	0.81
5	1.82	1.82	1.20	1.23	0.89	0.91
6	2.25	2.18	1.02	1.02	0.96	1.00
7	2.57	2.54	1.38	1.40	1.03	1.08
8th	3.09	2.91	1.27	1.23	1.19	1.15
9	3.49	3.27	1.58	1.56	1.21	1.22
10	3.63	3.63	1.38	1.40	1.27	1.29
11	4.25	4.00	1.73	1.70	1.36	1.35
12	4.19	4.36	1.52	1.56	1.38	1.41
13	4.65	4.72	1.78	1.83	1.37	1.47
14	5.33	5.09	1.69	1.70	1.46	1.52
15	? [3)]	5.45	?	1.95	?	1.57
16	6.06	5.81	1.86	1.83	1.65	1.63
17	6.17	6.18	2.05	2.07	1.64	1.68

18	7.09	6.54	2.05	1.95	1.78	1.73
19	7.22	6.90	2.21	2.18	1.80	1.77
20	7.66	7.27	2.11	2.07	1.85	1.82
30	10.06	10,90	2.27	2.57	2.14	2.23
50	17.87	18.17	3.25	3.35	2.63	2.88
100	37.87	36.34	4.87	4.77	4.64	4.07
500	178.17	181.69	10.42	10.74	9.00	9.09

2) [As in § 1 06, it was also interpolated with the addition of second differences.]

3) [Comp. the remark to Tab. I a.]

One would find the average agreement of the empirical values with the calculated ones very satisfactory. If, however, not inconsiderable deviations occur here and there, this can not be accidentally written in the careful revision of these values, but it is in the nature of things that among many accidental values calculated according to their law, they also happen to be stronger Deviations from the normal values occur. [Moreover, the relatively large deviations found among the values of the last four lines can be taken into account for the small *n of*them.]

[Taking into account, in addition to Tables I and II, the comparative table III, one finds the following principal values, comparable to each other, for the outcome of the true and the false means:

m	Q	**Q**	*U*	**U**	*V*	***V***
10	10.32	3.63	2,495	1.38	2.19	1.27
50	52.48	17.87	5.825	3.25	5.04	2.63
100	97.47	37.87	8.00	4.87	7.49	4.64

These show that the transition from true to false means, in fact, entails a reduction in the mean and probable differences, which is in sufficient accordance with the theoretically demanded reduction. It is namely:

m	$\mathbf{Q}^2 : Q^2$	**U** : V	***V*** : *V*
10	0.352	0.554	0,577
50	0.341	0.558	0.522

100	0.389	0.608	0,619

By contrast, the theoretical ratios are without taking into account the corrections for ***U*** and *U*, $\mathbf{Q}^2 : Q^2 = 0.363$; $\mathbf{U} : U = \mathbf{V} : V = 0.603$.]

It may be cited as a peculiarity that the value ***U***, which for the account of the false means holds, agrees closely with the simple average deviation from the *U* valid for the calculation of the true mean, or that ***U*** **is** close to ε [*U*], but only with such a large *m* that the correction ± 1.5 is no longer significant. This results both from the comparison of the formulas for both values:

$$\mathbf{U} = 0.48097$$

and 4) :

$$\varepsilon [U] = 0.48262 ,$$

as empirically confirmed for larger *m* .

[On the basis of the above compilation of the values of *U* and ***U*** in particular, ε [*U*] yields *m* = 10; 50; 100 resp. equal to 1.64; 3.44; 4.40. It is therefore in the same order ε [*U*] - ***U*** resp. equal to: 0.26; 0.19; - 0.47.]

Nor can one assure whether the numerical coefficient for both values can not really be assumed to be the same with equal advantage, since both coefficients derived in different ways and thereafter slightly different yield on both sides only approximate determinations and thus have no absolute validity.

4) [Comp. § 120 in the following chapter. Since, according to the definition given there, ε [*U*] = 0.60488 *U* and there on the other hand, neglecting the correction:

$$\mathbf{U} = U ,$$

it follows, on the basis of the correspondence of ε [*U*] and ***U***, that, as stated at that point, approximatively

0.60488 same

can be set.]

Probably the same relations extend to the other principal values, and the reported observational data give the opportunity to test it; but I have neglected to respond, partly in the expectation that the theory will take more possession of this process, partly in order not to extend the already extensive investigation.

Finally, here is the comparison of some distribution boards after calculation and experience.

Comparison of the observed numbers of *v* in the above tables with those calculated according to § 113 for some values of *m*.

v	$m = 4$		$m = 10$		$m = 20$		$m = 30$		$m = 50$	
	Obs.	calc.	Obs.	calc.	Obs.	calc.	Obs.	calc.	Obs.	calc.
0	1950	1779	494	480	176	174	94	95	49	44.5
2	1050	1182	588	581	256.5	267	169	159.5	84	80
4	-	38	112	128	130.5	121	90	93.5	51	57.5
6	-	-	6	10	33	32	36	38.5	32	33.5
8th	-	-	-	-	3	6	8th	13	14	16
10	-	-	-	-	1	-	3	0.5	8th	6
12	-	-	-	-	-	-	-	-	2	2
14	-	-	-	-	-	-	-	-	-	0.5

§116. [First Addition. The theoretical determination of the mean and probable value of v.]

[Each system of m positive or negative quantities $\Delta_1, \Delta_2 \ldots \Delta_m$ includes an average value Δ_0 and a difference value v, which indicates by how many the number v [5] of the values lying above Δ_0 the number μ exceeds the values below Δ_0. The values of $v = \boldsymbol{v} - \mu$ can therefore be any value of the series: $m - 2, m - 4 \ldots. 4 - m, 2 - m$, so that there are $m - 1$ positive or negative v values throughout, while the corresponding number of u values is $m + 1$. In this case, the case where a Δ_i ($i = 1, 2 \ldots m$) coincides with Δ_0 requires no special consideration, since it must be regarded as a limiting case in the case of the constant variability of these quantities which is either the case that Δ_i above Δ_0 or the case that Δ_i below $\Delta_0$is to add is. For example, for $m = 2$, the value of v *is* always equal to zero; for $m = 3$, on the other hand, v *is* either equal to + 1 or equal to - 1.]

[5] [v and μ replace here μ 'and μ .]

[On the other hand, for every $v = \boldsymbol{v} - \mu$, *there is* a manifold of systems $\Delta 1, \Delta 2 \ldots \Delta_m$ that can be determined as follows.]

[If Δ_0 δενοτεσ τηε mean varying between $-\infty$ and $+\infty$, then δ ρεπρεσεντσ a positive quantity that can assume all values from 0 to ∞ , and finally represent $\alpha_1, \alpha_2 \ldots \alpha_{\mu-1}; \beta_1, \beta_2 \ldots \beta_{\nu-1}$ independently of each other, the positive values from 0 to 1, we set:

$$\Delta_1 = \Delta_0 - (1 - \alpha_1)\,\delta$$

$$\Delta_2 = \Delta_0 - (1 - \alpha_2)\,\alpha_1\,\delta$$

$$, , , , , , , , , , , , , , ,$$

$$\Delta_{\mu-1} = \Delta_0 - (1 - \alpha_{\mu-1})\,\alpha_{\mu-2} \cdot , \alpha_1\,\delta$$

$$\Delta_\mu = \Delta_0 - \alpha_{\mu-1}\,\alpha_{\mu-2} \cdot , \alpha_1\,\delta \quad (5)$$

$$\Delta_{\mu+1} = \Delta_0 + (1 - \beta_1)\,\delta$$

$$\Delta_{\mu+2} = \Delta_0 + (1 - \beta_2)\,\beta_1\,\delta$$

$$, , , , , , , , , , , , , , ,$$

$$\Delta_{m-1} = \Delta_0 - (1 - \beta_{\nu-1})\,\beta_{\nu-2} \cdot , \beta_1\,\delta$$

$$\Delta_m = \Delta_0 - \beta_{\nu-1}\,\beta_{\nu-2} \cdot , \beta_1\,\delta .$$

First of all, all value systems $\Delta_1 \ldots \Delta_{m \text{ are obtained}}$ whose μ first values are below the respective mean value, while the last ν values exceed them. In fact, due to the fixed ranges of variability $\Delta_1, \Delta_2 \cdot , \Delta_\mu$ less than Δ_0; $\Delta_{\mu+1}, \Delta_{\mu+2} \cdot , , \Delta_{m \text{ is}}$ greater than Δ_0; so also is the sum of μ first Δequals $\mu\,\Delta_0 - \delta$ and the sum of ν last Δ equals $\nu\,\Delta_0 + \delta$, thus the sum of all Δ equals $m\,\Delta_0$.]

[Then all value systems $\Delta_1, \Delta_2 \cdot , \Delta_m$ to obtain, for which any of the values in the number μ below and the remaining ν are above the respective mean value is only necessary to all the possible permutations between the on the systems (5) μ first and the ν last Δ make what to $m! : (! M \nu)$ performs equations of the form (5), each of the same multiplicity of value systems $\Delta_1 \cdot , \Delta_m$ represents only with changed order of Δ each time , and whose association determines the total manifold of the value systems belonging to $\nu = \boldsymbol{v} - \mu$.]

[Let the Δ_i ($i = 1, \ldots, m$) be regarded as deviations from the true mean for which the GG holds. Then the W. for the occurrence of a single value is Δ equal to: .

It is also the W. for the occurrence of the system of the m values $\Delta_1 \cdot , , \Delta_m$ equals:

;

since, according to a well-known proposition of the theory of probability, W. is for the coincidence of several independent events equal to the product of W. for the occurrence of every single event. Finally, it is the W. for the occurrence of any system $\Delta_1 . , , \mathit{\Delta}_m$, which belongs to a well-defined, continuous manifold of such systems, is equal to:

$d\mathit{\Delta}_1 . , , d\mathit{\Delta}_m$ (6)

where the integral is to extend beyond the continuum of value systems in whose domain the existing value system is to fall. For the W. for any of a series of mutually exclusive events is, as the probability calculus teaches, equal to the sum of the W. of the individual events.]

[But according to the equations (5):

.

if for acronym (a ; β) =

(7a)

, (7b)

Thus one obtains as an expression for the W. that of m deviations $\Delta_1 . , , \mathit{\Delta}_m$ the μ first below, the ν last above the mean Δ_0 , the integral:

, (8th)

.

where to integrate over $\Delta 0$ from - ∞ to + ∞ , over δ from 0 to ∞ and over each of the α and β from 0 to 1. In agreement with this, the W. expresses that there are m deviations μ below and ν above the mean, that $\nu = \boldsymbol{v} - \mu$, from

.

, (9)

where the integral is to be taken between the same limits.]

[Since the integration over Δ_0 and over δ χαν βε χαρριεδ ουτ immediately by:

;

and for straight *m* :

;

for odd *m* :

;

for *W* [*v*] we obtain the simplified expression:

(10)

woselbst:

;

for straight *m* :

;

for odd *m* :

;

and where the integration for each α and β ισ to extend from the lower bound 0 to the upper bound 1.]

[The formula (10) first tries out in the simplest cases for *m* = 2 and 3, whose *W* [0] resp. *W* [1] is known from the outset. Namely, since for *m* = 2, *v* = 0, *W* [0] = 1 and, since *v* for *m* =3 is either equal to +1 or equal to -1, and both values are equally likely, *W* [+ 1] = *W* [- 1] = ½. And in fact one obtains from (10) for *m* = 2:

;

furthermore for *m* = 3:

.]

[From (10), by performing the integrations, the values of $W[v]$ are given for larger m. It should be noted that the sum of all $W[v]$ for a given m is 1, and that

$$W[+v] = W[-v], \quad (11)$$

since v goes into $-v$, μ interchanges with $\boldsymbol{v}$, which has no influence on the value of the integral.]

[Hereinafter one finds for $m = 4$:

; ;

;

,

It follows:

$$W[0] = 0.64908; \; W[+2] = W[-2] = 0.17546;$$

$$\boldsymbol{Q}^2 = 1.40368; \; \boldsymbol{U} = 0.70184.$$

Similarly, for $m = 5$,

$$W[+1] = W[-1] = 0.451075; \; W[+3] = W[-3] = 0.048925;$$

$$\boldsymbol{Q}^2 = 1.7828; \; \boldsymbol{U} = 1.1957.$$

For the two cases $m = 4$ and $m = 5$, the exact values for $\boldsymbol{Q}2$ and $\boldsymbol{U}$ **are** provided, whose comparison with the corresponding values of § 115 allows us to judge the reliability of the determinations there.]

[In this way, however, in the same way as in the previous chapter for the deviations from the true mean, formulas for $W[v]$ and thereafter those for $\boldsymbol{Q}^2$, $\boldsymbol{U}$, and $\boldsymbol{V}$ are obtained, which explicite the dependence of these values on m should the $(m - 2)$ -fold integral of (10) be in a generally valid form. However, such an execution, most conveniently from (9), can be gained by developing in series. However, since it leads to expansions, it is appropriate to set the value of $\boldsymbol{Q}^2$ and then - with the concession that such a gap remains unobjectionable to the objectives pursued here - ***U*** and ***V*** **are** derived on the premise that for large *m* the probability ratios *v are* governed by the GG. This assumption is admissible, since according to (11) the law of probability for *v is* symmetric with respect to the maximum value $v = 0$ *,* and furthermore the relations between $\boldsymbol{Q}^2$, ***U*** and ***V*** following from the GG, which are based on the formulas (1) to (4), have found a sufficient empirical test. In that case, however, a theoretical justification of the corrections given for ***U*** is waived.]

[The direct determination of $\boldsymbol{Q}^2$ can be achieved as follows. Note that for any system of deviations $\Delta_1 , \Delta_2 \ldots \Delta_m$ whose arithmetic mean Δ ισ $_0$, the difference $v = \boldsymbol{v} - \mu$ between the numbers of Δ_i (*i* =.) Lying above and below Δ_0 1, 2 .. *m*) can be represented by:

; (12)

as each quotient (Δ i - Δ 0) **:** is equal to 1 or equal to + - 1, depending on Δ *i* above or below Δ 0 is located. It is therefore:

, (13)

where the integration over each Δ_i *is* to extend from $-\infty$ to $+\infty$.]

[But now:

where the summation over all *i* and *k is to be extended* from the series of numbers from 1 to *m,* except the values $i = k$. It is therefore because

.

and all *m* (*m* - 1) integrals:

are equal to each other:

, (14)

where are the limits of integration, as stated above.]

[Now to evaluate the *m*- fold integral, set:

$$\Delta_1 = \Delta_0 + \delta_1$$
$$\Delta_2 = \Delta_0 + \delta_2$$

, , , , , , , , , , , , , , , , , , ,

(15)

Thus, in place of the Δ_1, Δ_2, Δ_m, which vary independently of each other between the limits $-\infty$ and $+\infty$, they also occur independently of one another between the same; Limits varying Δ_0, δ_1, δ_2 ... δ_{m-1}, and you get:

.

Where:

(16)

From this one wins by execution of the integration bez. Δ_0, δ_3, δ_4 . , δ_m :

(17)

But since $\delta_1 \delta_2$: = + 1, if δ_1 and δ_2 are simultaneously positive or negative, and the same quotient represents the value -1, if one of the two quantities δ_1 and δ_2 is positive, the other is negative, we obtain after simple transformations:

(18)

or, if ; :

(19)

Now is:

Consequently, finally, if $t_1^{\,2} = \tau_1$ and $t_2^{\,2} = \tau_2$ is set:

(20)

From this result, however, the sought value of $\boldsymbol{Q}$ 2 as represented by the formula (1) is obtained when quantities of the order 1 **:** *m are* neglected. By development to powers of 1 **:** *m* one obtains namely:

; (21)

thus in first approximation:

, (22)

From this it then immediately followed by the formulas (3) and (4) for U and V - but without the on U empirically found correction - when the G. G . for the probability ratios of v at large m *is* claimed.]

§117. [Second addition. Explanations on the empirical proof of the probabilities for Q, U and V by lottery lists.]

First of all, it might seem impossible at all to find a principle of empirical probation, since the formulas presuppose essential symmetry and validity of the GG of random deviations; but whatever object one tries to test, one can presuppose for the deviations from the means A neither the one nor the other condition from the outset as fulfilled. But you can artificially create an object that meets these conditions, according to the following principle.

Imagine, in order to explain the principle in the first conceivable way, in an urn a very large number, I mean 15,000 white and just as many black bullets, of which the first may count as positive, the last as negative deviations; but these spheres are to be described with positive and negative values of magnitude, each variable in such repetition, as corresponds to the W. of the corresponding error quantities according to the Basic Law. The correct mean value, from which the errors start, is the zero value. Now draw m balls and call positive sum $\sum\Delta'$ the sum obtained by multiplying each positive error size by the number of times it is drawn; accordingly with the negative sum $\sum\Delta_,$. If now $\sum\Delta'$ and $\sum\Delta_,$ by chance, are not found equal, the mean appears around ($\sum\Delta' - \sum\Delta_,$) : m, which value c *is* called, increased or decreased, depending on $\sum\Delta' > \sum\Delta_,$ or vice versa. The wrong mean is therefore equal to $0 \pm c$. If one has thus determined c , one can now count how many errors are greater and how much smaller than c and thereafter a $\pm$ ($\mu' - \mu_,$) orFind v for this case, and, having done $n$moves , find from this both a mean v *and* a probable v , which requires only an interpolation.

Now such a procedure with the urn and so many white and black bullets described in terms of sizes would be practically unfeasible; but you can keep the urn replaced by the lottery wheel, the white and black balls by even and odd numbers. Furthermore, in order to establish relationships among the 30,000 numbers which correspond to the probability ratios of the errors, all numbers from 1 to incl. 338 can be given the size 0.25, all from there to incl. 1015 the size 1, all from there to 1691 the size 2, all from there to 2366 the size 3 and so on and put this translation in a table, which at each Lotterienummer, which one meets in passing through the list, immediately gives information, which size it represents.

[This table is prepared by means of the t- table (§ 183), as follows. First of all, a decision has to be made as to which intervals the underlying t- values should progress to. In the interest of convenience, the interval 0.02, with the initial $t = 0.01$, is chosen. Since now the presumed number of lottery numbers, the same number of copies of a K.-G. 30,000, then the Φ corresponding to the interval limits areMultiply values by 30,000 to obtain, in their successive differences, the numbers of deviations that fall within the successive intervals. However, the deviations themselves are, as for our K.-G. consistently happens to think united in the middle of the interval in which they belong. Thus, since $t = \Delta : \varepsilon$, the first Δ ωουλδ βε equal to ε • 0.005; the second equals ε • 0.02; to set the third equal to ε • 0.04 etc; However, since the size of the mean deviation ε μαψ be arbitrarily set, then ε = 1 : 0.02 = 28.2095, after which the first Δ equals 0.25, the second Δ equals 1, the third equals 2, and so on. In order to finally secure this Δ the frequency of occurrence as required by the GG according to the *t*- table, each individual should be assigned as many lottery numbers as the number of associated deviations. This assignment could in itself be made quite arbitrarily, since each of the 30,000 numbers of the Wheel of Fortune has the same W. to be drawn. Of course, however, the natural order of the numbers is observed; Thus, the first Δ ιστhe first 338 numbers, the second Δ the following 677 numbers, etc. are attached, as indicated above, so that a table is drawn up, which reads as follows:]

size	number	size	number	size	number
0.25	1 - 338	14	8923-9548	47	24347 - 24626
1	339-1015	15	9549-10167		
2	1016-1691			74	28872 - 28946
3	1692 - 2365			75	28947 - 29018
4	2366-3038	25	15351 - 15877		
		26	15878 - 16393		
5	3039 - 3708	27	16394-16899	100	29854-29,865
		28	16900 - 17394		
10	6356 - 7005				
11	7006 - 7650			143	29998
12	7651 - 8289	45	23756 - 24056	150	29999
13	8290 - 8922	46	24057 - 24346	160	30000

Actually, of course, the deviations change continuously, whereas here each deviation quantity deviates from the following by 1; but this deviation interval is small enough in relation to the simple mean deviation, that is, after the given ratio 1 : 0.02 = 28.2095, to give a noticeably coinciding result with a continuous change in size.

I have now been offered Saxon lottery lists of 10 years, each of 32,000 to 34,000 numbers, of which I have left the numbers over 30,000 in the lists as not available at page. [From these 10 lists, the empirical data of Tables I and II above and hereafter the probabilities of the probability determinations of **Q** , **U** and **V were** obtained by the previous method .]

[It applies z. For example, the determination of *v* for *m* = 6. You then have to combine six consecutive numbers of the lists, ignoring the numbers above 30,000; Thus, if the numbers 28904, 24460, 32305, 16019, 157, 3708, 16928 are made, because it exceeds 30 000 for putting aside the 3-th, and the remaining six in the above table in deviation sizes Δ implement the are positive for even numbers, negative for odd numbers. Thus, the designated numbers represent the quantities + 74, + 47, - 26, - 0.25, + 5, + 28 with the mean + 21.3; consequently, with respect to the latter, $\mu' = \mu$, = 3 and $v = 0$. This determination, carried out 2000 times, gave the values listed in Tab. I, b under *m* = 6, *n* = 2000.]

XVII. The simple and two-sided Gaussian law.

§ 118. If even the simple GG, which we have explained § 24 - 29, because of the generally in K.-G. presuppose asymmetric W of the collective deviations. *A* not directly on K.-G. is applicable, the two-column GG (§ 33) must be claimed, according to which all the provisions of the simple Basic Law on K.-G. become transferable, if one takes the deviations from *D*instead of *A* and those after simple GG together for both sides bez. *A* valid values ± Δ , *m* , η = ΣΔ **:** *m* ρελ . each page in particular resp. through ∂ ', **m**', *E'*
= ∂ ' : **m** ' and ∂ ; **m** , , *e* , = ∂ , **: m**, replaced. With this in mind we go to those already in v. given information about the simple GG, which are to be assumed here, nor to the following additions the same one.

It has already been stated that the distribution tables of the GG which have been carried out up to now, ie the Φ panel and the φ panel, do not have any Δ **:** η , for which they were given § 27, but bez. Δ **:** η , short *t, are* set up. Such a table is communicated in the appendix (§ 183).

In the same way, the fundamental Gaussian determination is based on the fact that the W. or the relative number of a single value ± Δ ισ short a certain quantity, equal to:

, (1)

wherein , .

In order to have them between given limits of Δ, one has to multiply the previous expression by $d\,\Delta$ and take the integral of this between the respective limits; gives generally:

(2)

or after replacing h by $1 : \eta$, Δ by η t, $d\,\Delta$ by η dt:

(3)

and the W. or relative number of Δ between $t = \Delta : \eta$ $= 0$ and a given t is hereafter:

, short = $\Phi\,[\,t\,]$. (4)

This probability $\Phi\,[\,t\,]$ is now expressed for the different values t by the table given in the appendix. In order to have the absolute number of Δ between the limits $t = 0$ and a given t, one has to multiply $\Phi\,[\,t\,]$ by the total number m .

As we all know, the integral expression for $\Phi\,[\,t\,]$ can not be integrated in finite form, but it can be represented in the following infinite series, which converges strongly and is therefore useful for the calculation of Φ , as $t = \Delta : \eta$ less than 1, hence $\Delta < \eta$, di <1.772 45 · η is:

(5)

Since the Φ always following bez. t *are* taken, the addition $[\,t\,]$ can be ignored. All powers of t are positive, because $t = \Delta : \eta$, Δ and η are both positive and negative.

Now it is important to note that if, as is often the case in our applications, the value Δ , which enters $t = \Delta : \eta$, is very small against the mean error η , hence t itself is very small, all the members of the series (5) can be neglected against the first; according to which:

(6)

, (7)

But in this neglect of the higher terms, in the view of (5), the value Φ ισ determined to be a little too big, and so we have to set more precisely:

, (8th)

where ω ισ a very small positive value. From (8) but follows:

, (9)

whereafter t neglecting ω is found a little too small, di according to the approximate values (7).

§ 119. According to the GG, the value η has certain normal relationships to some other values derivable from the distribution tables, insofar as they are subject to the GG, whose confirmation is the more likely to be expected the more *m increases* .

Let q = the root of the mean square of deviation, which is considered by the astronomers to be the mean deviation par excellence, and w the so-called probable deviation, ie the deviation, which, if one takes positive and negative deviations both according to absolute values, just as many larger deviations as a smaller one above itself, that is to say basically the central value of the deviations, not to be confused with our central value par excellence, which is denoted by C , in that it is not a deviation Δ but an a . You now have the following normal relationships:

$$= 1.253\,314\ \xi\ \eta\text{ , so markedly} = {}^{5}/_{4}\,\eta\ ;$$

$$= 0.797\,885\ \xi\ q\text{ , thus markedly} = {}^{4}/_{5}\,q\ ;\ (10)$$

$$q = 1.482\,604 \cdot w\ ;\ w = 0.674\,489 \cdot q$$

$$\eta = 1.182\,947 \cdot w\ ;\ w = 0.845\,347 \cdot \eta$$

By substituting the previous expressions for η in $t = \Delta : \eta$, one can also set without changing the associated Φ :

or t = , (11)

From this it appears at first indifferent to which expression for t keep to. But it is not entirely υνιμπορταντ whether one first determines q from the squares of the deviations, $\sum\Delta^2$, in order to find η or w *by* means of the previous formulas, or conversely η or w from the simple deviations, from one of these values But the direct determination of q from the squares of deviations has a somewhat greater certainty than that of η as a means of simple deviations, and the latter a not inconsiderably greater than that of w by counting the deviations, which translates to the values derived from the above formulas. Therefore, in the physical and astronomical theory of measurement one likes to hold to the value $t = \Delta : q$, after direct determination of q from the squares of the deviations; but also obtain the same certainty by applying the other expressions for t , if η or w *has* been derived from the directly determined q in the above formulas , whereas the certainty is less if one η or even w in the expression of $t$determined directly from the simple deviations, and nothing is gained by applying the expression $t = \Delta : q$, if q is derived therefrom by the use of previous formulas from the directly determined η or w .

Although the use of the value $t = \Delta : q$, after the direct determination of q , has a fundamental advantage of security over the other modes of t , in the collective theory we generally prefer the value $t = \Delta : \eta$ after direct determination of η from AΔ use, because with the large amount of discrepancies with which we are dealing, in general, in this Maßlehre, squaring them would be too cumbersome, the advantage of the safety in use of the right given $q$however, before the directly determined η, ιτ is only insignificant, and in the case of large m , it loses its significance appreciably. In fact, while the probable error of the directly determined q equal

is the one of the directly determined η and that of the

directly determined w immediately

[1)].

[1)] [The derivation of these probable errors is given by GAUSS in the Zeitschrift fur Astronomie vol. I (Works, Vol. IV, pp. 116, 117) and in the treatise on the method of the least squares (Berliner Astron. Jahrbuch für 1834, p 293 and 298). It should be noted that the numerical value for *w,* which is found at the indicated position in GAUSS, is disfigured.]

§ 120. All the above are known things. But it may not be without interest to add a few sentences derived from the GG.

One must beware of χονφυσινγ the sum of the deviation σθυαρεσ $\Sigma\Delta^2$ with the squares of the deviation sum $(\Sigma\Delta)^2$. Now, if one takes the trouble of oβταινινγ , besides the latter, values which are easy to obtain by squaring $\Sigma\Delta$, and the former, by determining the squares of deviation, one can take into account that $(\Sigma\Delta)^2 = (m\,\eta)^2$ and $\Sigma\Delta^2 = mq^2$, from the equation:

easily the interesting equation:

, (12)

or, if you call the phrase on the left side P,

$$P = \pi \quad (12a)$$

deriving that the sum of the deviation squares multiplied by 2 *m,* ie twice the deviation number, divided by the square of the deviation sum, is equal to the circle ρατιο π . For a moment, the formula may be called the *P* formula.

On the other hand, according to the previous formula, the sum of the squares of deviation which can be calculated directly from the more easily determinable squares of the deviation sum according to the formula is obtained with great difficulty:

, (13)

except that the directly determined sum $\Sigma\Delta^2$ is determined to be somewhat safer than that derived from $(\Sigma\Delta)^2$ according to the previous formula .

To the two middle δεφεχτσ , the simple $\eta = \Sigma\Delta : m$ and square , can be a third

(14)

which I will call the circle mean error and which, according to the above expression, is obtained by dividing the sum of the squares of deviation by the sum of the deviations, or, which comes to the same thing, the square of the mean square error with the simple mean error.

I give it the above name because it represents a turning point in the following sense in relation to the circle ratio π expressed by the *P*- equation . If we first assume that the equation is exactly satisfied by the existing deviations, then in the case that deviations greater than $\eta\pi$ grow, *P is* greater than π ; however, *P* becomes smaller than π when deviations smaller than $\eta\pi$ grow. The change is proportional to the distance of the relevant deviation from $\eta\pi$. The proof of this I pass over [2)] .

2) [It follows that *P* in its dependence on any single deviation values Δ_i, reaches its minimum when or = $\eta\pi$ .. At the same time it is evident that *P* reaches its absolute minimum with the values 2, when each of $\Delta_i = \eta\pi$ becomes.]

I have found the *P*- equation of many pure errors of the psychophysical method of average errors to be admirable.

According to the given expressions, the three center errors have the following relation:

, (15)

and it can be shown that the deviation sums above these mean errors amount to the total sum of the deviations according to Chap. XVIII have the following relations, where *e,* as always, is the fundamental number of natural logarithms:

= 0.72738 rel. η ; = 0.60653 rel. q ;

= 0.45594 bez. $\eta\pi$;

of which the first two values are very close to the ratio 7 : 6.

The corresponding ratio of the lower deviation sums is, of course, obtained by subtracting previous numbers from 1, and then it turns out that the lower and upper deviation sum are related to each other. q behaves very close like 2 : 3.

With respect to w , the respective ratio of the upper variance is 0.79655; however, the value with respect to which the upper deviation sum is equal to the lower one is 1,17741 · q .

The upper deviation numbers have the following ratios to the total number of deviations:

0.42494 rel. η ; 0.31731 bez. q ; 0.21009 bez. $\eta\pi$; 0,5 bez. w ;

according to which these ratios for w , η , q, $\eta\pi$ *αρε* very close to 5 : 4 : 3 : 2.

Nor is it possible to define, as an average deviation of second order, the mean to be denoted by η_2 from the differences of the individual Δ from the mean η οφ τηε same, that is [if$\sum\Delta''$ the sum and μ'' the number of Δ which are smaller than η , respectively $\sum\Delta''$ and μ'' δενοτε the sum and number of Δ which are greater than η , such that $\mu''\eta - \sum\Delta'' = \sum\Delta'' - \mu''\eta = \frac{1}{2} m\,\eta_2$]:

, (16)

approximatively with

provoking.

Just as one can represent the value π by a function of the deviations according to GG, so also the value *e.* If, according to the above statement, the deviation sum above q divided by the total deviation sum is the same , conversely the total deviation sum divided by the upper bez. q and the quotient squared equals e .

§ 121. All the preceding propositions concerning the GG presuppose, to their full validity, a large, strictly speaking, infinite number of deviations from which the respective quantities are derived, which, as has already been stated, does not hinder the fact that even a very moderate number Number of deviations is a very approximate empirical confirmation of the previous sentences; and there for the successful treatment of a K.-G. in any case a not inconsiderable number m of copies a and consequently deviations of the same from both sides of *D*Thus, not only [after replacing the simple GG with the two-column] one can expect, but also find, a very approximate confirmation of the previous sentences. In the meantime the deviations from the so-called true values, that is, those which follow from an infinite m , or so-called errors, which, depending on the size of the finite m on both sides and of the m' and$m$, after each side in particular still remain, deserve after all, essential attention; and it refers partly to the so-called probable errors, partly to the corrections of the determination from finite *m*the errors change the true value indifferently and randomly into positive or negative or in a certain direction increase or decrease by a value dependent on the size of the m [3)].

[3)] [The corrections for the mean deviation values were described in § 44 u. 45 communicated; the probable errors for η , *q,* and *w* can be found in § 119 above. Also worth mentioning is the probable error, which is to be expected from the determination of the arithmetic mean *A* from *m* values, and which is to be set *w* : *w,* as usual, the probable error di the probable deviation of the individual values (see above under (10)).

§ 122. [In order to prove the validity of the bilateral GG in comparison with the distribution law of the K.-G. Based on panels I and III of chapter VIII, comparison tables between the observed and calculated *z* values are to be prepared. Those panels are suitable for such comparisons, since they have a weak asymmetry, and thus give the expectation that an advantage offered by the application of the two-sided law will be more strongly manifested in the event of greater asymmetry.]

[From the 5 reduction positions of Table I (§ 64) I choose position $E_{,}$ = 368, and from the 4 reduction positions of Plate III (§ 65), position $E_{,}$ = 60 with the remark that the former is the relatively weakest, the latter has the relatively strongest asymmetry in comparison with the other layers. For both panels are now both with respect to A , the values $t = \Delta : \eta$ and thereafter $\Phi [t]$ and with reference to D_p, the values $t' = \partial' : e'$ and $t_{,} = \partial_{,} : e_{,}$ and thereafter $\Phi [t']$ and $\Phi [t_{,}]$ is calculated, where the Δ , ∂' , $\partial_{,}$ of A or D_p up to the respective interval limits $a \pm \frac{1}{2}i$ (not up to the a self) extend. Then, the differences of the successive Φ values, which are to be referred to as φ values, are formed, and the found $\varphi[t]$ with $\frac{1}{2}m$, the $\varphi [t']$ resp. $\varphi [t_{,}]$ with m' resp. $m_{,}$ multiplied. In this way, the z values calculated according to the simple and the two-sided GG result in comparison with the observed tabular values in the following two tables. Here the numerical values of η , e' and $e_{,}$ without correction, are taken as basis, since the affixing of them with the size of the m and the desired degree of accuracy is irrelevant:

Comparison of the empirical *z* of panel I (vertical circumference of the skull) with the theoretical after single and two-sided GG

E = 1 mm; i= 5; A= 408.2; D_p =409.7; η= 11.1; e'= 10.4; $e_{,}$= 11.9; m = 450; m'= 210; $m_{,}$ = 240.

a	empirical *z*	Theoretical *z*		difference	
		Ref. *A*	bez. D_p	bez. *A*	Ref. D_p
363	-	0.5	0.5	+ 0.5	+ 0.5
368	1	1	1	0	0
373	2	3	3	+ 1	+ 1
378	5	6	7	+ 1	+ 2
383	17	13	13	- 4	- 4
388	24	22.5	22.5	- 1.5	- 1.5
393	36	35.5	34.5	- 0.5	- 1.5
398	41	49	47	+ 8	+ 6
403	59	60	58	+ 1	- 1

408	65	64	64	- 1	- 1
413	65	60	62	- 5	- 3
418	51	50	52	- 1	+ 1
423	40	37	38	- 3	- 2
428	17	24	24	+ 7	+ 7
433	19	13	13	- 6	- 6
438	4	7	6	+ 3	+ 2
443	2	3	3	+ 1	+ 1
448	2	1	1	- 1	- 1
453	-	0.5	0.5	+ 0.5	+ 0.5
total	450	450	450	46	42

Comparison of the empirical *z* of panel III (recruits) with the theoretical ones after single and two-sided GG

$\boldsymbol{E}$ =1 inch; i =1, A= 71.75; D_p =71.99; η= 2.04; e'= 1.92; $e_,$= 2.16; m= 2047; $\boldsymbol{m}'$= 963.5; $\boldsymbol{m}_,$ = 1083.5.

a	empirical *z*	theoretical *z*		difference	
		bez. *A*	bez. D_p	bez. *A*	bez. D_p
60	1	-	-	- 1	- 1
61	0	-	-	0	0
62	0	-	0.5	0	+ 0.5
63	0	1	1.5	+ 1	+ 1.5
64	2	3.5	4	+ 1.5	+ 2
65	15.5	10	12	- 5.5	- 3,5
66	26	26	28	0	+ 2
67	54	58	59	+ 4	+ 5
68	108	110	108	+ 2	0
69	172	179	174	+ 7	+ 2

70	253	252	243	- 1	- 10
71	290	304	298	+ 14	+ 8
72	330.5	315	318	- 15.5	- 12.5
73	296	282	291	- 14	- 5
74	223.5	217	226	- 6.5	+ 2.5
75	142	143	145.5	+ 1	+ 3.5
76	75	81	80.5	+ 6	+5.5
77	38	40	37	+ 2	- 1
78	13	17	15	+ 4	+ 2
79	3.5	6	5	+ 2.5	+ 1.5
80	2	2	1	0	- 1
81	1	0.5	-	- 0.5	- 1
82	0.5	-	-	- 0.5	- 0.5
83	0.5	-	-	- 0.5	- 0.5
total	2047	2047	2047	90	72

As you can see, in both tables the total of deviations between observed and calculated values, taken in absolute terms, is smaller for the two-sided law than for the simple one, even if the difference is insignificant, especially for the first comparison table. But what matters more is the greater faithfulness attained by the two-sided law in comparison to the simple one in the representation of the core of both panels, opposite the end divisions.]

[Incidentally, the comparison of the *z*- values of the two-sided law with the corresponding *z* -values of the simple law in both cases consistently shows that from the center of the panel for growing *a* those first larger and then smaller, for decreasing *a* those smaller first and then larger as these are. The reason for this lies in the two panels of the common direction of asymmetry, and these ratios would reverse if the asymmetry were reversed.]

XVIII. The summation law and the supplementary procedure.

§ 123. So far, as far as I know, the GG has been used merely to determine the relative or absolute number of deviations Δ of *A* between given limits of deviation; but formulas for the relative and absolute sum of the deviations of *A* between given limits of deviation can be developed in connection therewith and, as it were, as a corollary of them . of the GG in general, as long as they remain valid and mutually applicable for the mutual deviations, as a symmetrical W of deviations. *A* consists; in the case of asymmetric W but again after the two-column GG their validity for each side in particular claim, if the deviations bez. *D* instead of bez. *A* accepts, and *m* , AΔ , η , *t* for each side, in particular by relative ***m*** **,** , ∂ **,** , *e* **,** , *t* **,** and ***m*'** , ∂ ', *e* ', *t'* replaced.

But the results in relation to the sum of the deviations deserve all the more attention because they do not share the disadvantage of the results with respect to the number of deviations, only by an integral or series that can not be reduced to a finite expression, hereafter tabulated since they can be expressed in a finite form, moreover, they can become important through the supplementary procedure (§128) which makes them possible. The following applies, in accordance with the procedure to be explained below.

§ 124. In order to determine the sum of the deviations up to a certain limit of deviation from the densest values to one side, let us say the positive, that is, to the limit ∂ ', of which the corresponding applies to the negative side, take the total sum of Deviations from this side, ιε ∑∂ ', form from this the simple mean deviation *e'* = ∑∂ ' : ***m*** ', suppose *t* = ∂ ' : *e'* , then, down below, form the value exp [- t^2], then the absolute sum of the deviations from ∂ '=0 to the given ∂ *'is* equal to: ∂∂ ' (1 - exp [- t^2]) and beyond ∂ ' to ∞ equally: ∑∂ ' · exp [- t^2]; the proportional sum up to ∂ *',* but the previous absolute, divided by the total sum ∑∂ ', which is denoted by *T* , equal to 1 - exp [- t^2], beyond that exp [- t^2].

Instead of determining the absolute and relative sum up to a certain limit ∂ ' and beyond, this determination may also be made up to a certain number of deviations, which are *z'* , as long as there is a large ***m*'** , as assumed here , *z* **' :** ***m*** *'can be found* after the previously determined *t* and vice versa as Φ in the *t* - table. So do *z* **' :** ***m*** *'* given, they were looking in the *t* - table, the *t* and use it in the previous way to the sum determination.

Insofar as each value *a* in the *a* column of the distribution table actually represents a whole interval *i* in which the *z* values written on *a* are distributed, which we call the perimeter interval of the relevant *a* , then the limit to which we are the sum How to take the number of deviations, not by an *a of* the *a* - column itself, but by the boundary of its perimeter interval, whereby it adjoins the perimeter interval of the adjacent *a* , to be regarded as definite.

Instead of determining the sum up to given limits of *D* on each side, it is possible to determine them also between any limits on each side in the same way as the number on each side, by subtracting from each other the sums belonging to the limits according to the former mode of determination.

§ 125. To find exp [- t^2], add 2 log t to 0,63778 - 1, look for the number in the logarithmic tables, take it negative, that is, subtract it from the next larger integer and add it at the back with a negative sign added; search for the number again, this is exp [- t^2].

This calculation has to, of course, no difficulty, however, is seen to be a bit cumbersome, and in order to spare for each case, however, you can then for equidistant $t = \Delta : \eta$ or to the multiplication of η with to spare , for those of Δ **:** η the corresponding values of

and then specify 1 - exp [- t^2] and take the equidistant values close enough to interpolate between them. This is followed by such a table, the values of which, of course, should be closer to one another to allow a very precise interpolation.

Table on the deviation sums from Δ to ∞ , the total sum as unit ,

	exp [- t^2]		exp [- t^2]		exp [- t^2]
0.00	1.00000	1.00	0.72738	2.00	0.27992
0.05	0.99920	1.05	0.70403	2.05	0.26245
0.10	0.99682	1.10	0.68035	2.10	0.24568
0.15	0.99286	1.15	0.65641	2.15	0.22961
0.20	0.98735	1.20	0.63232	2.20	0.21425
0.25	0.98030	1.25	0.60813	2.25	0.19960
0.30	0.97176	1.30	0.58395	2.30	0.18566
0.35	0.96176	1.35	0.55983	2.35	0.17241
0.40	0.95034	1.40	0.53586	2.40	0.15986
0.45	0.93757	1.45	0.51210	2.45	0.14798
0.50	0.92350	1.50	0.48861	2.50	0.13677
0.55	0.90820	1.55	0.46545	2.55	0.12621
0.60	0.89173	1.60	0.44270	2.60	0.11628
0.65	0.87417	1.65	0.42038	2.65	0.10696

0.70	0.85558	1.70	0.39855	2.70	0.09823
0.75	0.83606	1.75	0.37726	2.75	0.09006
0.80	0.81569	1.80	0.35654	2.80	0.08245
0.85	0.79455	1.85	0.33641	2.85	0.07536
0.90	0.77273	1.90	0.31692	2.90	0.06877
0.95	0.75031	1.95	0.29809	2.95	0.06266

	$\exp[-t^2]$		$\exp[-t^2]$		$\exp[-t^2]$
3.00	0.05700	4.00	0.00614	5.00	0.00035
3.05	0.05176	4.05	0.00540	5.05	0.00030
3.10	0.04694	4.10	0.00474	5.10	0.00025
3.15	0.04249	4.15	0.00416	5.15	0.00022
3.20	0.03841	4.20	0.00364	5.20	0.00018
3.25	0.03466	4.25	0.00318	5.25	0.00015
3.30	0.03123	4.30	0.00278	5.30	0.00013
3.35	0.02809	4.35	0.00242	5:35	0.00011
3.40	0.02523	4.40	0.00211	5.40	0.00009
3:45	0.02263	4.45	0.00183	5.45	0.00008
3.50	0.02026	4.50	0.00159	5.50	0.00007
3.55	0.01811	4.55	0.00137	5.55	0.00006
3.60	0.01616	4.60	0.00119	5.60	0.00005
3.65	0.01440	4.65	0.00103	5.65	0.00004
3.70	0.01281	4.70	0.00088	5.70	0.00003
3.75	0.01138	4.75	0.00076	5.75	0.00003
3.80	0.01009	4.80	0.00065	5.80	0.00002
3.85	0.00893	4.85	0.00056	5.85	0.00002
3; 90	0.00790	4.90	0.00048	5.90	0.00002
3.95	0.00697	4.95	0.00041	5.95	0.00001

				6.00	0.00001
				6.15	0.00001
				6.20	0.00000

§ 126. The derivation of the sum law as a function of *A* after a simple GG is this.

According to the simple GG, the absolute number of deviations between $t = 0$ and a given value of $t = \Delta : \eta$ is taken together on both sides :

; shortly m Φ [*t*]. (1)

To have the associated sum, one has to multiply previous value below the integral sign by Δ , which gives:

, (2)

But since $t = \Delta : \eta$ and thus $\Delta = t\,\eta$, one has to substitute this value for Δ in the previous integral:

, (3)

The general integral of 2 τ t exp [- t^2] *dt* is, considering that $tdt = d\ t^2$, integrable in finite form, namely equal to - exp [- t^2] and hence between the limits $t = 0$ and $t = t$ equals (1 - exp [- t^2]), which multiplies by $m\ \eta = \Sigma\Delta$, gives:

$$\Sigma\Delta\,(1 - \exp[- t^2]), \quad (4)$$

as the sum of Δ between $t = 0$ and a given *t.*

Be short

$$1 - \exp[- t^2] = T \quad (5)$$

set, so is

$$\Sigma\Delta \cdot T \quad (6)$$

the required value.

Now it is expressed in infinite series:

, (7)

it acquires in a very small *t* di $\Delta : \eta$ is sufficient to maintain the first two terms, which for very small *t* noticeable are:

$$\Sigma\Delta \cdot T = t^2 \cdot \Sigma\Delta . \quad (8th)$$

In the case of asymmetry one has to start from D instead of A and apply the two-column GG, d, i. instead Aδ put $\Re\partial'$ or $\Re\partial,$ and t on each side as well by e' or $e,$ to make them dependent as before on η.

§ 127. In order to compare observation with calculation, it is of course necessary to determine the deviation sum itself up to given limits. Now, for the empirical determination of the total $\sum\partial$ oφ each page (according to § 74):

$$\partial, = \boldsymbol{m}, D - \sum a,;$$

$$\partial' = \sum a' - \boldsymbol{m}'D; \quad (9)$$

Formulas that change for the determination up to the given limit $\partial,$ or ∂' *of* each side merely insofar as under $\boldsymbol{m},$ and $\boldsymbol{m}'$ no longer the totality of the deviation numbers of each page, but only the deviation numbers up to the relevant boundary, and under $\sum a,$ ' $\sum a'$ *is* not the totality of a each side, but again only to the given limit to understand, according to which we denote the values in question with two dashes below and above, instead of in terms of totality with a little stick. If now *D*In general, falling into a certain interval, the part of $\boldsymbol{m}''$, $\boldsymbol{m}''$, $\sum a''$, $\sum a''$ which falls within that interval, as previously stated (§ 72 and 73), is to be determined by interpolation; remaining part is given by the observation itself.

Let us explain this on Plate I of the 450 Skulls. [For the reduction position $E, =$ 368 (§ 64), $D_p = 409.7$ falls within the interval 405.5 - 410.5. It is thus $a_0 =$ 408; $z_0 = 65$; $i = 5$; $g_1 = 405.5$; $x = 4.2$, and we obtain for the D_p to the first interval boundary 405.5 reaching $\sum\partial''$, di for $y\,D_p - Y$, where y *is* the number and *Y*indicates the sum of the intervention interval, according to formulas (13) and (8) of IX. chapter:

$$y = \square \cdot 65 = 55;\ Y = 55 \cdot 407.6;\ yD_p - Y = 55 \xi 2.1 = 116.$$

Accordingly, the following comparison table between theory and experience is obtained for the lower deviation sums of Table I:

Comparison of the empirical $\sum\partial''$ with the theoretical for panel I (vertical χιρχυμφερενχε οφ the skull).

$\boldsymbol{E} = 1$ mm; i = 5; $D_p = 409.7$; $e, = 11.9$; $\sum\partial, = 2840$.

∂''	$\sum\partial''$		difference	$\sum\partial'' : \sum\partial,$		difference
	empir.	theor.		empir.	Theor.	
0 to 4,2	116	111	- 5	0,041	0,039	- 0,002
"9,2	511	491	- 20	0,180	0.173	- 0.007

"14.2	991	1034	+43	0.349	0.364	+ 0.015
"19.2	1592	1599	+ 7	0.561	0.563	+ 0,002
"24.2	2113	2079	- 34	0.744	0.732	- 0.012
"29.2	2566	2423	- 143	0.904	0.853	- 0.051
"34.2	2725	2636	- 89	0.960	0.928	- 0.032
"39.2	2798	2749	- 50	0.982	0.968	- 0.014
"44.2	2840	2806	- 34	1,000	0.988	- 0.012

From this it can be seen with what approximation the absolute and relative sums of deviation as given by the tablet are represented by the law of summation. It should be remembered that the empirical values are subject to a uniform distribution of the *a* resp. ∂ ωερε δετερμινεδ within the individual intervals, while the theoretical calculation is based on the assumption that the distribution also corresponds to the GG within the intervals.]

§ 128. Addition. The supplementary procedure.

If, as is generally customary, in a distribution panel only the total number, but not the total sum of *a,* which falls above and below a certain value, is given just the prefix *v* and *n* , but not the preamble *V* and the sum *N* , then Although *C* , but neither *A* nor D_p can be obtained directly, nor the deviation functions with respect to these values, no distributional calculation will be possible. In the meantime one can do this according to the following, admittedly somewhat laborious, procedure, which I call the supplementary procedure.

Rather, rather than D_p , $D_{i \text{ is determined}}$, which as a rule deviates little from D_p $_{in}$ order to be substituted for it. First, consideration is given to *v, V, n, N* , but the still incomplete deviation numbers **m** " , **m** " and deviation συμσ ∑∂ " , ∑∂ " according to known sharp method only from the executed parts of the panel. But one also determines the total deviation numbers **m ,** = **m** " + *v* and **m'** = **m** " + *n* , hereafter *v:* **m ,** and *n :* **m** '. In the following table we can find values α whose method of calculation is given below, but through the table should, at least for some values, be the trouble of . calculation be spared the table is just to small values *v:* **m ,** and *n:* **m '** extended, as it is in far most cases only to those, where the table is not enough, must α be calculated directly.

Afterwards one finds the full sum of the lower and upper deviations of D_i as follows:

, . (10)

Hereinafter [1] :

, ;

, (11)

[1] [Since this presupposed validity of the two-column GG respect to D_i the existence of the proportional law: $e' : e_, = m' : m_,$ a result, it is possible with regard to the fact instead of the above, applicable without regard to this law formula also directly: $A = D_i + e' - e_,$ which gives an indication of the security of the determination compared to the derivation of A above .]

Some of the numerical values $v : m_,$, $n : m'$ associated sum fractional values α of the deviations of each side with respect to *D*.

	α
.1626	0.37726
0.1105	0.27992
0.0726	0.19960
0.0461	0.13677
0.0282	0.09006
0.0167	0.05700
0.0095	0.03466
0.0052	0.02026
0.0028	0.01138
0.0014	0.00614
0.0007	0.00319
0.0003	0.00159
0.0002	0.00076
0.0001	0.00035

The calculation of α so happens: One seeking to $m' : m_,$ or to $m' : m'$, whichever it is the negative or positive side as $\Phi[t]$ taken, the value t and take $\alpha = \exp[-t^2]$.

This method of determination depends on the fact that, for each side of the deviations from D_i, the simple GG is considered valid after the number and average deviation found for this page, in short the modified G G for the totality, and depends on that in the following Activation developed principle.

[The three values: 1) the relative number of deviations, 2) the relative sum of the deviations, 3) the quotient of the deviation itself, up to which of D_i the relative number and sum are determined, and the mean deviation , are so dependent on each other that any two can be calculated from the third. It is because of the GG for the deviations of a page, for example, the positive:

; ; ; (12)

where **m'** and $\Sigma\partial'$ *represent* the total number and sum of deviations of this page, ∂'' but the deviation means up to which the incomplete number **m''** and the incomplete sum $\Sigma\partial''$ *are*extended. It can therefore, in the manner indicated above, to **m'**: **m'** resp. **M'**: **m**, through the intermediary of *t* the value ∂' ': ∂' resp. $\Sigma\partial''$: $\Sigma\partial$, calculated therefrom when ∂''resp. ∂'' is found empirically ∂' resp. $\Sigma\partial$, determined according to (10).]

To illustrate this determination by a specific example, in QUETELET's table of French recruits [2] $v = 28\,620$; $n = 2490$; $m = 100{,}000$. [Now we find $D_i = 1.6273$ m, so **m**, = 55 951; **m'**= 44,049; **m''**: **m**, = 0.488848; **m''**: **m'**= 0.94347; hereinafter from the *t* - Table first if $t = 0.46420$ and $1 - \exp[-t^2] = 0.19385$; second, $t = 1.34843$ and $1 - \exp[-t^2] = 0.83769$. Consequently, one obtains from (10) the total συμ $\Sigma\partial$, = 3740,5; ∂'= 2410.7 as $\Re\partial''$ = 725.1 and $\Re\partial''$ = 2019.4. Finally, (11) gives e, = 0.0669; e'= 0.0547; $A = 1.6140$. It is thus $D - A = 0.0133$, while $e, - e' = 0.0122$; both values should be equal to each other, but their divergence is due to the fact that the initial value D_i is proportional to the determined $D_p$something deviates. Quettel himself, who by appraising comparison of the observed probability values with the theoretical values of his probability table, arrives at a distribution board, says: "la waistle moyenne est de 1.62 m environ".]

[2] [Lettres sur la théorie des probabilités, p. 401. "Waist of conscrits francais".]

One might think that even in cases where there is a complete series, the observed values become abnormally too small downwards, as is the case with the Leipziger and Annaberger recruits, only the supplementary procedure to the higher part of the series. but which is still on the same side of D, need to apply to a $\Sigma\partial$, which is uninvolved in the influence of the abnormality downwards, or as if the normal ratio between the number and magnitude of the deviations, which is assumed to be higher, also reached the lower end. But this is not the case; on the contrary, one can only expect a useful result of the supplementary procedure in so far as the lower part of the series, which is excluded from the calculation, which is *b* , is just as normal as the one taken in the calculation, which is called *a* . In fact, we assume that the relative number of deviations from a certain deviation value to the end, that is, in part *b,* is too great, so the relative number above it, in part *a,* be abnormally too small; but in the supplementary procedure one assumes that it is normal, which contradicts itself. Therefore, even if one proceeds according to the supplementary procedure in such abnormal series, one comes to absurd conclusions. Of course, in such series, by the supplementary μετηοδ , the directly obtained value $\sum$,, and the value of *A increases.* - So I took the part of the Leipzigers as *a* negative side, which ranges from *D* = 69.71 to 66.5, as *b* the part from there to the end, whereby one can remember (according to § 15), that 66 is the value below which the underneath falls. The value derived from the totality of $\sum\partial$, 9935, the 9097 derived from the Supplementary method, was noticeably the same as the value of $\sum\partial = 9070$, which follows from the positive parts of the series considered normal. The value of *A* derived directly from the totality of the series was 69.62, and the 69.70 obtained by the supplementary method, that is, the value *D, was* remarkably equal. Now if but *D* really the average, so the median would have to coincide with it, so $m' = m$, be, whereas m , = 4257; m' = 4145.

XIX. The asymmetry laws.

§ 129. [In the two preceding chapters the GG has been developed so far that it is ready for use as a suitable instrument for the distributional calculation of K.-G., as well as in essential symmetry, as well as in essential asymmetry of deviations. Now experience has shown that indeed the GAUSSian law of error represents the correct law of distribution with a small fluctuation of the individual values about its mean value, and that even in weak asymmetry, in which it remains doubtful, whether only a disturbance of essential symmetry or essential asymmetry If the two-sided law grants advantages to simple laws, then the two-sided GG can be regarded as the sufficiently effective distribution law of the K.-G. set up with a weak relative fluctuation. This Basic Law of Distribution for K.-G. then relies only on the experience and requires no theoretical justification. From the empirical point of view, therefore, it remains only the task to derive the special laws of an essentially asymmetrical distribution, which were already purportedly pre-recorded (in v. Chapter), as consequences of the Basic Law.

[Even though this Basic Law is sufficiently supported by experience, it is certainly interesting to have theoretical assumptions concerning the K.-G. to theoretically justify the two-sided GG in a similar way as it did for the simple law in error theory. This should be done after deriving the special laws in the Supplement to this chapter.]

§ 130. [The special laws of essentially asymmetrical distribution are divided into two groups. The first contains provisions of the initial value, according to which the latter

1. is the densest value, ie has the maximum z,
2. has the property pronounced in the proportional law.

The second group gives relations between the principal values, the arithmetic mean A, the central value C, and the densest value D, insofar as the distances of these values and their relative position are theoretically determined and properties of the numbers of deviations belonging to A and D are developed 1).]

1) [In addition to these laws, § 33 also lists the extreme laws. However, they are valid for symmetry as well as asymmetry of the deviation values, and thus are not laws of substantially asymmetrical distribution. Moreover, since they give rise to more detailed discussions, they will receive special treatment in the following chapter.]

[To derive these laws, the two-sided GG is to be used as the basis for the distribution of the copies of a K.-G.

, (1)

Here, as usual, m' and $m_{,}$, the numbers of deviations above and below the initial value D, ∂' and $\partial_{,}$ the distances of the deviations of D, h' and h taken from their absolute values, finally mean the reciprocal values of e' and $e_{,}$, where e' and $e_{,}$, the average values of ∂' and $\partial_{,}$, are. But it should be the output value Dnot from the outset as the closest value nor as the value determined by the law of proportionality, since both properties are first to be proved. On the contrary, D *should be* regarded as an initially arbitrarily chosen initial value, which can only be proven by law (1) as having the value attached to those two properties. It is still to be noted that ζ 'and $\zeta_{,}$ denote no numbers, but in geometrical interpretation only those to ∂' resp. $\partial_{,}$, present as abscissas, on the latter perpendicular ordinates of the distribution law imagine. In contrast, the numbers of deviations always refer to intervals and are represented by area strips, so that the equations

$$z' = \zeta' \, d\partial' ; z_{,} = \zeta_{,} \, d\partial_{,} \quad (2)$$

indicate how many deviations according to the law (1) between the infinite bounds ∂' and $\partial' + d\partial'$ resp. $\partial_,$ and $\partial_, + d\partial_,$ to the interval enclosed by the latter of the size $d\partial'$ resp. $d\partial_,$ fall. Accordingly, the determined W. W' and $W_,$ that a deviation between the specified limits can be found. It is made by:

(3)

designated.]

[By equations (1), for each finite value of ∂' and $\partial_,$ the associated value of ζ' and $\zeta_,$ and thus also the associated value of z' and $z_,$ or of W' and $W_,$ is uniquely determined , However, for the output value itself, to which the deviation values $\partial' = 0$ and $\partial_, = 0$ belong, this uniqueness is missing, unless

h

$'m' = h_, \; m_,$ or (4)

Because it will be for this value:

; (5)

so that an uninterrupted transition of the two curves representing the equations (1) takes place in fact only if the conditional equation (4) is fulfilled. But that this conditional equation must necessarily be fulfilled is clear from the following consideration.]

[It goes without saying that an interval of given size and given position can only belong to a certain number of deviations. The consequence of this is that the same number must also be given to an infinitely small interval, which is to be regarded as the limit of a finite interval, even if it may be regarded as the boundary of an interval extending in the upper or in the lower part of the distribution table. But for the initial value ζ' *is* different from $\zeta_,$ Also, the number of deviations for the interval associated with the output value depends on whether the latter is thought to be on the side of the above or below the deviations located below the output value. Since this is not permissible, then $\zeta' = \zeta_,$ and thus the conditional equation (4) must be fulfilled.]

[It would be insignificant to counter this by attaining unambiguousness for the numbers but not for the differences. For the probability determinations (3) relate to each side of the deviations in particular, without taking the other side into account or being affected by it. If one wants to have a determination of W. which is considered jointly by both sides , then the same must refer to the total number $m = m' + m_,$ the deviations, and then it must be stated:

, (6)

so that, as it should be, for $\partial' = \partial_{,} = 0$ the uniqueness of the probability determination is given on the basis of (4).]

[The conditional equation (4) must therefore be included in the distribution law (1). But this is the fulfillment of the proportional law of the initial values

$$e' : e_{,} = \mathbf{m}' : \mathbf{m}_{,} \quad (7)$$

required. At the same time, this value manifests itself as the closest value, since both ζ' and $\zeta_{,}$ for the zero value of the deviation θυαντιτψ ∂' and $\partial_{,}$ reach the maximum.]

[To illustrate this distribution law, the two following curves may serve, the first of which shows the course of the values above D, with the indication of the probable and average deviations $w = DW$; $e' = DE'$; $q = DQ$; the second shows the course of the values on both sides of D with the two main values A and C next to D and the two simple mean deviations $e' = DE'$; $e_{,} = DE_{,}$ in mind.

It should be noted that the ordinates introduce relative values by the place of the values of ζ' and $\zeta_{,}$ of the formula (1) by $2h'\mathbf{m}' = 2h_{,}\mathbf{m}_{,}$ divided values $\zeta' : 2h'\mathbf{m}'$ and $\zeta_{,} : 2h_{,}\mathbf{m}_{,}$ are set. Further, $h' = 1$; $h_{,} = 2/3$ was adopted. Therefore, the maximum value is DB in the two curves, equal to $1 :$; further:

$$e' : e_{,} = 2 : 3;\ e' = 0.564;\ e_{,} = 0.846;\ D - A = 0.282;$$

$$D - C = 0.222;$$

The unit of measurement is 5.6 cm for the first curve, 3.2 cm for the second.]

§ 131. [Only exceptionally will the numbers $\mathbf{m}'$ and $\mathbf{m}_{,}$ the deviations above and below the initial value *D be* equal to each other. In this exceptional case the central value are *C* and the arithmetic mean of *A* with *D* united. For it is $\mathbf{m}' = \mathbf{m}_{,}$ so that the condition characterizing the central value is satisfied; from the equality of $\mathbf{m}'$ and $\mathbf{m}_{,}$ but continues to follow, on the basis of the law of proportionality, that $e' = e_{,}$ and hence $\mathbf{m}' E' = \mathbf{m}_{,} e_{,}$. This implies that the mutual variance sums are also equal to each other, thereby determining the arithmetic mean.]

[However, is how to assume a rule, ***m*' ** of ***m* ,** different, the two main values are *A* and *C* Never *D* together, and it can be their distances from *D* from GG derived as follows.]

[Denote the larger of the two numbers ***m*' ** and ***m* ,** by ***m*",** the smaller one by ***m*"** and mark the values ∂ , *e* , *h* and *t* lying on the side of the ***m*"** in accordance with the provisions made earlier (§ 33) then the central value *C* is to be looked for as the value which in association with *D* delimits an interval containing ½ (***m*"- m"**) deviations, for it is:

, (8th)

so that above and below the value determined for the species, there are the same number of deviations as is required for the central value. However, it follows from the law of distribution that γ = *C* - *D* indicates the distance between the values *C* and *D* regardless of their mutual position:

, (9)

or, if *h* " ∂ ' '= *t; h*" γ = *t "is* set:

, (10)

It is thus considered that *h* " = *e* " ,

C - *D* = γ = *t "e"* , (11)

where either γ ισ to be calculated directly from (9) or determined to be *t 'by* means of the *t*- table on the basis of (10) as the value too short of Φ " .]

[The distance *C* - *D* is thus essentially dependent on the quotient (***m*"- m"**) **:** ***m*"** . If the latter is equal to zero, then γ also becomes zero, and *C* coincides with *D* , as already noted . If, however, this quotient is not equal to zero, but is sufficiently small that its second power can be neglected, it is permissible to approximate Φ [*t* "] as the size of the same order by:

or (12)

represent and thus:

(13)

or:

(14)

to put. On the other hand, $C - D$ *reaches* the maximum possible value if $(\boldsymbol{m}'' - \mathrm{m}'') : \boldsymbol{m}''$assumes the value 1, ie if $\boldsymbol{m}'' = 0$ and $\boldsymbol{m}'' = m$, ie if the totality of deviations is on one and the same side of the initial value As a result, the asymmetry becomes infinitely large, and in this limit, (10) the simpler equation:

, (15)

so that $t'' = w : e''$, where w *represents* the probable value of the deviations, which should be set equal to $0.845347 \cdot e''$ after § 119. For the distance $C - D$ one obtains the equation:

$$C - D = w = 0.845347 \cdot e''.] \quad (16)$$

[In the general case (11) as well as in the two limiting cases (14) and (16) this determination of $C - D$ is based on the two - sided GG as distribution law. It is therefore the empirical determination of this distance in a proposed distribution table, the easiest way to directly calculate C and A *by* means of equation (26) or (29) of XI. In general, a value deviating from the theoretical determination found here results. It is different with respect to the distance A-D between the arithmetic mean A and the output value D, since the formulation of the formulas for this distance is based only on the properties of A and D , which are also the basis of the empirical calculation, while there is no reason to use the GG.]

[If one considers namely that the greater of the two deviation sums ∂' and $\wp\partial$, as a result of the proportional law on the same side of D is found on which the greater of the two deviation numbers, namely $\boldsymbol{m}''$, is to be sought, after which the larger of both sums by $\sum\partial''$, the smaller one by $\sum\partial''$ is called, so one can put:

$$\sum\partial'' = \sum a'' - \boldsymbol{m}''D$$

$$\sum\partial'' = \boldsymbol{m}''D - \sum a'' \quad (17)$$

From this follows by subtraction:

$$\sum\partial'' - \sum\partial'' = \sum a'' + \sum a'' - (\boldsymbol{m}'' + \boldsymbol{m}'')D = \sum a - mD ,$$

and, after division with m , one obtains that:

.

the equation:

, (18)

which, however , does not yet take into account the property of *D of* satisfying the proportional law. For this purpose, put in (18):

$$\Sigma\partial '' = \boldsymbol{m}'' e'' ; \Sigma\partial '' = \boldsymbol{m}'' e''$$

or, what is the same, since $\boldsymbol{m}'' = \mathbf{m} - \mathbf{m}''$ and $\boldsymbol{m}'' = m - \boldsymbol{m}''$:

$$\Sigma\partial '' = me'' - \boldsymbol{m}'' e'' ; \Sigma\partial '' = me'' - \boldsymbol{m}'' e'' .$$

This leads to the equation:

, (19)

in which according to the proportional laws:

$$\boldsymbol{m}'' e'' - \boldsymbol{m}'' e'' = 0$$

is, so that finally:

$$A - D = e'' - e'' \quad (20)$$

results in a relationship that already in the XI. Cape. when it came to the exploitation of the properties of D_p in the interest of its determination from the empirically given tabular values.]

[Since according to the proportional laws:

$$e'' - e'' = (\boldsymbol{m}'' - \mathbf{m}'') ,$$

so equation (20) can also be in the form:

(21)

or, if as above:

is put into the form:

$$A - D = 2\ \Phi '' \cdot e'' \quad (22)$$

to be brought.]

[The determination of the distance *A - D* is therefore indeed independent of the existence of the GG, so that for each distribution table the equation (20) must exist, if otherwise *A is*calculated as the mean and *D* as D_p, ie, according to the proportional law have been.]

[The limits can also be specified for *A - D.* If $\boldsymbol{m}'' = \boldsymbol{m}''$, it follows from (21) that also *A* = D, in accordance with the remark already made, according to which *C* and *A* coincide with *D* at the same time . If, however, $\boldsymbol{m}' = m$ and $\boldsymbol{m}'' = 0$, the asymmetry is therefore infinite, so is

$$A - D = e'', (23)$$

ie equal to the simple mean deviation, while according to (16) *C - D represents* the probable deviation. Further, in the event that $(m'' - m'') : m''$ ισ a small quantity whose second power may be neglected, formulas (12), (13), and (14) come into force, so that 21) or (22) the equation:

(24)

can be derived.]

§ 132. [On the basis of the above determination of the distances *C - D* and *A - D* , *A - C* can also be found as the difference of the two previous distances, after which the laws of distance for the three main values *A, C* and *D* can be given in the following form:

1) for quite arbitrary values m'' and m'' , ie for an arbitrary degree of asymmetry, one has according to the formulas (11) and (20) resp. (22):

$$C - D = t''e''$$

$$A - D = e'' - e'' = 2\,\Phi'' \cdot e'' \quad (25)$$

$$A - C = (A - D) - (C - D) = (2\,\Phi'' - t'')\, e'';$$

2) for $m'' = 0$ and $m'' = \mathbf{m}$, ie for the case of infinite asymmetry, relations (16) and (23) exist; it is thus:

$$C - D = 0.845347 \cdot e''$$

$$A - D = e'' \quad (26)$$

$$A - C = 0.154653 \cdot e'';$$

3) if $(m'' - m'') : m''$ *introduces* a small quantity whose second power can be neglected, that is, if the asymmetry is very small, one can put according to formulas (14) and (24):

; (27)

4) in the event that there is no asymmetry, in which case $m' = m$, **then** finally:

$$C - D = 0$$

$$A - D = 0 \quad (28)$$

$$A - C = 0 .$$

It should be noted that, although, as the derivation of the variances for A - D and C - D can immediately recognize A and C at the same time on the side of the m'' are, but that only the absolute values of these distances are determined, and It therefore remains to be seen whether A and C differ in the positive or in the negative direction from D. The former is the case when $m' > m_,$; the latter, if $m_, > m'$.]

§ 133. [From these laws of distance the distance relations and in particular the π - laws can be obtained by division. You get:

1) for the general case in which the degree of asymmetry is not subject to any condition:

(29)

;

2) in the case of very weak asymmetry:

(30)

;

3) in the case of infinite asymmetry:

(31)

,

The values reported under 2) and 3) represent the limits between which the provisions applicable to the general case vary. In particular, the relations which are valid for weak asymmetry are of interest, since with the small fluctuation of the copies of the K.-G. so common that it can be called a rule. For this reason, the relations (30) are given a special name and are called the π - laws.]

[Of the three quotients, the first one is usually taken into account and therefore, for the sake of simplicity, a special one. Letters, namely denoted by p . Thus, p or (C - D) **:** (A - D) is expected to become not less than 0.785 and not more than 0.845, unless irregularities disturb the course of the empirical values of a distribution panel and the correspondence with the theory that solely for the above provisions.]

§ 134. [That C and A lie on the same side of D has already been noted; but that C lies between A and D is clear from the following statement.]

[After formula (29) is in general:

, (32)

where t'' *is* the value associated with Φ'' in the t -table. Now note that Φ'' can only represent values between 0 and ½, since

.

so a look at the t -table teaches that consistently

$$t'' < \Phi'', \quad (33)$$

because only from the value $\Phi = 0.6209$ are the three-digit t values greater than the associated Φ values, in order to remain larger until the end of the table. In addition, because:

$$<2$$

and thus more:

$$t'' \quad < 2\Phi'',$$

in fact:

$$C - D < A - D. \quad (34)$$

This law, according to which C always lies between A and D , is called the law of position.]

[The position law has the consequence that the asymmetry of deviations bez. D the opposite sign has as the deviations bez. A. Namely, since with respect to C , the mutual deviation numbers are equal to each other, there is for each value above C the inequality $m' < m$, and for each value below C the inequality $m' > m$, . It is thus, if A *is* above C ,

$$\mu' < \mu\,, \text{ ie } \mu' - \mu\,, \text{ negative.}$$

But then *D* lies below *C,* so that:

$$m' > m_,\text{, ie } m' - m_, \text{ is positive.}$$

Conversely, if *A* below and *D* above *C* is located. This reversal of the asymmetry with respect to *A* and *D* is called the inversion law , which is therefore an outgrowth of the law of position.]

[Additive. The theoretical foundation of the bilateral GAUSS law.]

§ 135. [So far, the two-sided GG was based on experience as the sufficiently probable probability law of K.-G. established. If, in addition to the empirical proof, we still want a theoretical foundation of this law, hypotheses must be given as to the K.-G. which allow a derivation of that law. The establishment of such hypotheses finds its justification in that they lead to the law to be derived and contain it as if in the germ. And even if experience alone decides the correctness of the established law, the insight into the nature of K.-G. promoted.]

[First of all, I prove that it suffices to presuppose the value D_p determined by the proportional law as the most probable value in order to derive the two-sided GG in the same way as in the error theory the simple GG is based on the assumption that the arithmetic mean is that most likely value, inferred. The hypothesis of the arithmetic mean in error theory is thus hypothesized in collective theory that the proportional law is the most probable value among the specimens of a K.-G. determine, completely equal to the side.]

[To prove this, suppose that *m* specimens *a of* a K.-G. for which there exists a value $D_p = a_0$ determined according to the proportional law . There are then $m_,$ values of *a,*namely $a_1, a_2, , a_3$.... below D_p and m' values *a* , namely a', a'', a'''..., above D_p, and it orders for the deviations of these values of $D_p = a_0$ according to the proportional laws the equation:

or, if the lower deviations by ∂_1, ∂_2 . , , the upper ones are denoted by ∂', ∂'', .

$$m'^2 \partial_, + m'^2 \partial_2 + \cdots + m_,^2 \partial' + m_,^2 \partial'' + \cdots = 0. \quad (35)$$

Let W be the deviations $\partial_1, \partial_2 \cdots$
$\partial', \partial'' \cdots$ through $\varphi(\partial_1), \varphi(\partial_2) \cdots \varphi(\partial'), \varphi(\partial'') \cdots$. Then the W. for the coincidence of all *m* deviations by the product of the *m* W., thus by:

expressed.]

[But since a_0 is supposed to represent the most probable value according to the underlying hypothesis, according to the known principles of probability theory, the product of W. must be greater for the deviations of the presented values *a* from a_0 than for the deviations from any one other values other *than* a_0. It must therefore

to be a maximum. Now, for the sake of brevity:

so is thus:

(36)

to put.]

[This equation must be the same as equation (35). Therefore, do you bring (36) into the form:

so lucid that:

, (37)

where *k is* an arbitrary constant. Out:

but follows

and from this by integration:

, (38)

At the same time one recognizes that *k has to introduce* a negative value if $\varphi(\partial)$ is to reach its maximum for $\partial = 0$.]

[It is thus for the below $D = a_0$ located deviations that are now indiscriminately by $\partial ,$ will be referred to:

, (39)

where $c ,$ is an even closer constant and - $h ,^2 = ½ k \mathbf{m}'^2$. On the other hand, for the deviations above $D = a_0$, which may be represented indiscriminately by ∂', one finds:

, (40)

where again the determination of c 'is still outstanding, while - $h'^2 =$ ½$k \mathbf{m},^2$]

[Finally , to determine the constants c' and $c ,$ the W. is that of the $\mathbf{m}'$ upper and $\mathbf{m} ,$ lower deviations, any one between 0 and ∞ is, as it goes without saying, equal to 1. It must therefore:

and:

his. This leads to:

.

to:

, (41)

Therefore, finally:

(42)

with the following condition from the given values for h' and $h ,$

.] (42a)

§ 136. [In this justification of the two-sided GG, it may be perceived as a defect that the underlying hypothesis of the Proportional Law is inferior in simplicity and evidence to the hypothesis of the arithmetic mean in error theory. For, in the first place, one can seek support for it only in experience, as it has been described in § 42 as a fundamental fact of experience, that the K.-G. allow the determination of a densest value which coincides sufficiently closely with the values defined by the law of proportionality.]

[It is therefore of interest that another hypothesis can be made, based on simple and obvious considerations about the origin of the K.-G. supports. For the time being it leads to a uniform distribution law; however, since the latter permits the determination of a densest value which approximately satisfies the law of proportionality, the two-sided GG also presents itself as an approximation to that uniform law. This leads to the realization that the division into two parts of the law of distribution, as determined by the use of the law of distribution GG is conditional, not by the nature of the K.-G. but it may well be motivated by the need to make available the law which follows from the hypothesis to be set up for a convenient use satisfying the requirements of collective measurement.]

[In order to make clear the essential points in the development of this hypothesis, first, contrary to the actual circumstances, a K.-G. provided that its specimens differ only in a small number of equidistant and finite gradations in size. For example, five size levels may exist, and the sizes themselves in turn equal to:

$$a, a + i, a + 2 i, a + 3i, a + 4i \quad (43)$$

his. Then it is natural to attribute the difference in the size of the games special forces, each of which in the case of their activity to increase *i* generated. One will therefore assume four forces K_1, K_2, K_3, K_4, the way that each can act as well as not act. If none of the four forces comes into action, a specimen of size *a is produced* ; If only one of the four forces acts, then the specimen is given the size $a + i$; but if two, three or all four forces act, then the quantity $a + 2i$, $a + 3i$ or $a + 4i$ generated. From the W., which stands for the effectiveness of each individual force, then the frequency of the occurrence of the copies of a certain size step will depend and thereby the distribution law be conditional. Is obtained namely when the forces independently of each other with the W. p_1, p_2, p_3, p_4 act and accordingly, the W for the lack of its effect by $q_1 = 1 - p_1$, $q_2 = 1 - p_2$, $q_3 = 1 - p_3$, $q_4 = 1 - p_4$, the following representations for the W. of the different size stages:

$W[a] = q_1 q_2 q_3 q_4$;

$W[a + i] = p_1 q_2 q_3 q_4 + q_1 p_2 q_3 q_4 + q_1 q_2 p_3 q_4 + q_1 q_2 q_3 p_4$;

$W[a+2i]$
$= p_1 p_2 q_3 q_4 + p_1 q_2 p_3 q_4 + p_1 q_2 q_3 p_4 + q_1 p_2 p_3 q_4 + q_1 p_2 q_3 p_4 + q_1 q_2 p_3 p_4;$

$W[a+3i] = p_1 p_2 p_3 q_4 + p_1 p_2 q_3 p_4 + p_1 q_2 p_3 p_4 + q_1 p_2 p_3 p_4;$

$W[a+4i] = p_1 p_2 p_3 p_4.$ (44)

It can be seen that a symmetrical distribution of the copies on the different size levels only possible if z. B. $p_1 + p_3 = p_2 + p_4 = 1$, or if the occurrence of the effect of each force the same *W.* as for the absence of the effect of one of the other forces. Than it will be:

$W[a] = p_1 p_2 q_1 q_2$

$W[a+i] = (p_1 p_2 + q_1 q_2)(p_1 q_2 + p_2 q_1)$

$W[a+2i] = (p_1 p_2 + q_1 q_2) 2 (p_1 q_2 + p_2 q_1) 2 - 2 p_1 p_2 q_1 q_2$

$W[a+3i] = (p_1 p_2 + q_1 q_2)(p_1 q_2 + p_2 q_1)$

$W[a+4i] = p_1 p_2 q_1 q_2.$

Any other determination of W. leads to an asymmetrical distribution of the specimens on the different size stages. For example, 1. For $p_1 = p_2 = p_3 = p_4 = p$, 2. For $p_1 = p_2 = p_3 = ½$, $p_4 = p$, where p and $q = 1 - p$ *are* different from ½ are:

1. 2.

$W[a]$

$= q^4 \; {}^1/_8 q$

$W[a+i] =$

$4pq^3 \; {}^1/_8 (3q + p)$

$W[a+2i] =$

$6p^2 q^2 \; {}^1/_8 (3q + 3p)$

$W[a+3i] =$

$4p^3 q \quad {}^1/_8 (q + 3p)$

$W[a+4i]$

$= p^4 \; {}^1/_8 p$

One can thus always specify other asymmetric distributions as specializations of the general scheme (44), while only in the above way a symmetrical distribution is possible. But each of them is based in the same way on the assumption that four independent from each other forces are present, each of which has a certain W. for their effect and in the case of their work to increase in size *i* generated.]

[However, in reality there is no K.-G. which allows only five magnitudes, separated by finite and constant intervals, to be distinguished. Rather, the specimens are steadily distributed over the size range bounded by the extreme values, so that even by increasing the size stages, where then instead of five a larger number would have to be chosen, nothing wins. But the size range which the copies of the K.-G. constantly satisfy, divide into intervals of constant size *i* and determine the interval size of the kind, that within each individual interval the distribution of the copies may be assumed as uniform and the distribution law as constant. This is the case when *i*introduces a small size whose second power may be neglected compared to finite magnitudes. Then it is also permissible to think of the specimens falling on the interval as united in the middle of the interval, so that in this way we are led back to the idea of the size stages with constant intervals. The initial conception, however, is now modified insofar as the specimens no longer belong to the individual size steps themselves, but to the corresponding intervals, and the size steps serve only as representatives of the intervals.]

[Taking this modification into account, the size range of the examples of K.-G. can now be replaced by an indefinitely large number of size steps, so that the variables themselves appear

$$a,\ a + i,\ a + 2\ i,\ ....\ a + ni \quad (45)$$

are representable. It is therefore only necessary, instead of the limited number of four forces selected in the above example, to assume an indeterminate number *n of* such forces, and to assign each one a certain value for its operation, in order to determine for each size step a value as determined above. and thus to obtain a certain distribution of specimens across the whole size range. At the same time it is clear that this distribution is symmetric only if the *n* forces can be combined in pairs and for each pair whose W. are equal to p_i and p_k, $p_i + p_k = 1$. Any other determination of this W. leads to an asymmetric distribution. But if the latter can be pursued in its lawfulness, then no arbitrarily chosen force must be randomly assigned to each acting force. For the sake of practicability of the mathematical treatment of every force, it may therefore be attributed to the same W. for their coming into effect.]

[This leads to the following hypothesis:

1) There is an indefinite number *n* of forces [2)]

$$K_1, K_2, \cdots K_n$$

provided that they participate independently in the production of the copies of a K.-G.

2) There exists the W. p for the occurrence and the W. $q = 1 - p$ for the absence of the effect of each individual force.

3) Every force produces, in the case of its action, the increment i, where i *introduces* such a small quantity that its second power, in addition to finite magnitudes, may be neglected.]

2) [The term "forces" is chosen for brevity only; it may be understood as including all the peculiarities, of whatever kind, which have a changing influence on the size of the copies of a K.-G. are able to exercise.]

[Hereinafter, a specimen, in whose creation none of the n forces participate, receives the quantity a, whose W. $W[a] = q^n$, while on the occurrence of all forces the quantity $a + ni$arises, for which $W[a + ni] = p^n$ is. But if a force x participates in a single instance, its size becomes $a + xi$; and since

different systems of each x forces can be formed, for each system but the W.

$$p^x \cdot q^{n-x}$$

exists, so is:

, (46)

Now for large n, x and $n - x$ the formulas are:

,

With regard to this, one obtains:

(47)

Assuming here pn and qn as integers, we assume that n is divisible by the common denominator of the p and q fractions, so that the generality of the subsequent evolution is not limited, then instead of x and n x with advantage $pn + x$ and $qn - x$ write, where now x has to go through all positive numbers from 0 to $+ nq$ and all negative numbers from 0 to $- np$; at the same time, $a + xi$ is $a + pni + xi$ or, if$a + pni$ *is* briefly *denoted* by a_0, to be replaced by $a_0 + xi$. One finds thus:

(48)

From this one wins with consideration that:

;

following form of representation:

(49)

It is valid as long as *x: pn* and *x: qn are* smaller than 1.]

[If this law is to represent W. for the finite values of the deviations *xi* of a_0, then *x* must be assumed to be the order of magnitude 1 **:** *i* . On the other hand, *n is* a higher-order quantity if the extreme deviations *pni* and *qni* are very large in comparison with the values *xi* considered . This is indeed true, since the extreme deviations increase with the number of copies on both sides and thus, from the point of view of the theory, are to be assumed to be growing indefinitely. Let *n* be a size of order 1 **:** i^2 provided. Then the quotient x^2 **:** *n represents* a finite quantity and the quotient *x: n* in the same way as the quotient x^3 **:** n^2 represents a quantity of the order *i.* Thus, if quantities of the order i^2 and higher order in the series representation of φand ψ αρε neglected, one can put the law of probability (49) in the following simple form:

.

or:

, (50)

if $xi = \Delta$ and $ni^2 = k$ *is* set.]

[In the derivation of this law, it was assumed that the copies of the K.-G. in the centers $a_0 + xi$ the intervals represented by the series of values (45) may be thought of in unison. In reality, however, the specimens are steadily distributed within the intervals, so that the probability function is to be assumed as a continuous function of the deviations Δ whose integrals between the limits of the intervals are given by the $W[a_0 + \Delta]$. If, therefore, the probability function is denoted by $w[a_0 + \Delta]$ then:

$$W[a_0 + \Delta] = \int w \cdot d\Delta,$$

or with regard to the smallness of i :

$$= w \cdot i.$$

One finds therefore first for the interval centers:

; (51)

but since w *is* a continuous function of Δ , this representation has to hold for any Δ.]

[Hereinafter, by differentiation, one finds the maximum value of w from the equation:

;

or (bearing in mind that a part of w does not disappear, on the other hand Δ here is to be neglected a size of order i, and consequently $i\Delta^2$):

,

Thus, the densest value D falls on:

,

If this value is chosen as the initial value for the law of probability, then $a_0 = D + \frac{1}{2}i(q - p)$; Finally, if $w[D + \partial]$ is replaced by $\varphi(\partial)$, then $\Delta = \partial - \frac{1}{2}i(q - p)$

(52)

as the final form of the law to be derived.]

[Now it is still a matter of proof that the initial value *D* approximately satisfies the proportional law on the basis of the law (52). For this purpose:

set so that:

, (53)

Now, if ***m*** ′indicates the number above *D* and *m* indicates the total number of deviations:

,

Accordingly, for the below *D* preferred number ***m*** , :

,

Is referred to further above and below *D* located sums of the deviations by ∂ ′and ℘∂ , so is:

,

One finds from it:

; , (54)

Thus:

= , if $\beta = \frac{3}{4} \pi\alpha = 2.356\ \alpha$. (55)

In first approximation one can therefore

$$\alpha = 1;\ \beta = 2$$

set, so that in fact approximatively:

, (55a)

as the proportional law requires.]

[But if the proportional law is valid, then the two-sided GG can replace the uniform probability law (52) with a corresponding approximation. The same is to be presupposed in the form (6), which refers to the mutual deviations, since the law (52) also takes into account the upper and lower deviations. So be it:

, (56)

Here is due to the calculated deviation numbers and deviation sums:

, (56a)

However, since the approximate validity of the proportional law requires that $\frac{3}{4}\pi$ is rounded down to the integer value 2, it is also $\frac{1}{2}\pi$ and $4/3$ to be regarded as equivalent and

(56b)

to put; Also, with the same authority in the representation of h' and h , instead of $\frac{1}{2}\pi - {}^{2}/_{3}$ as well $\frac{1}{4}\pi$ and ${}^{2}/_{3}$ are set].

[The replacement of the unitary law (52) by the bilateral GG therefore results in replacing the member the Member

occurs, which receives a positive sign for positive ∂, a negative sign for negative ∂.]

[Both (52) and (56) represent for $p = q$ the simple GG, which is thus developed as a special case together with those general laws from the established hypothesis. If the latter this case adapted from the outset, it is not significantly different from the hypothesis that HAGEN [3)] to derive the simple has set G. G. for error theory.]

[3)] [Broad probability, Berlin, 1837. 34. - The hypothesis HAGEN's reads: "The error in the results of a measurement is the algebraic sum of an infinite number of elementary mistakes that all are equal, and each of which can be just as positive as negative. ".]

[It should be noted that the asymmetry here is represented by quantities of the order i . It therefore becomes infinitely small when i becomes infinitely small. In the above derivation, however, *i was* not assumed to be infinitely small, but only so small as to allow i^2 to be neglected against finite magnitudes.]

[It should be noted that for the uniform law of probability, instead of the densest value *D,* a value other than the initial value can be chosen. In the form of representation (51), for example, it is the arithmetic mean value which is made the starting point of the deviations. With respect to $a_{0,}$ the sums of the mutual deviations are equal to each other, so that $a_{0\ \mathrm{is}}$ in fact the arithmetic mean *A.*]

XX. The extreme laws.

§ 137. Among the usually taken into account elements of a K.-G. belong to the extreme values offered by the distribution table of the same, ie the measure of the largest and smallest specimen; Also, it has a multiple interest to deal with it. For mere curiosity, one may be interested in the size of the largest giant and the smallest dwarf in a given country, or even at all, which is the greatest heat or cold, up to which the temperature has risen in a given place But the statement of the extreme values of an object under investigation also has a scientific value for the knowledge of the same, by contributing to the characteristic of it, in view of the number of specimens under which these extremes are observed; and the expectation as to which limits a future specimen will be seeking, beyond which it does not presumably rise, under which it will not sink, may sometimes become practical. Thus, the highest expected level of a river can determine the height of the protective dam or the height of plants on its shores, the greatest cold to be expected limits the planting of certain plants, etc

It should not be forgotten that the size of the extremes depends on the number of specimens subject to observation; If, for example, the height of a river does not exceed a certain extent within 100 years, it can not be expected that it should not be the case in a thousand years, as this offers greater latitude for the development of the extremes, which immediately generates interest it seems reasonable to find a law of the dependence of the greatness of the extremes on the number of specimens, an interest which is at the same time scientific with the practical. Immediately every empirical determination of the extremes has significance only for the number of specimens from which the determination has taken place; But it can also serve as an empirical document for the general determination of extremes with changed numbers.

So far, this point has been overlooked several times, in more than one place, by the magnitude of the absolute or relative difference between the extremes: $E' - E,$ or $(E' - E,) : A,$ which consists of different *m* at different K. -G. used to compare the absolute or relative variability of the subject matter used, which may have quite erroneous implications.

Hereby the Apercu seems to be based on the fact that, if one determines only the extremes of a large number, one can count on preserving, if not the absolutely possible extremes, but those which are very close to them, and in the absence of other persuasions could be content with the found ones. But this assumption of an approximately attainable limit of extremes with increasing *m* has neither empirical nor theoretical in isolation; but it is true only from both points of view, that the greatness of the extremes grows in much smaller proportions than the size of the *m* , but, when *m is* thought to increase to infinity, it always grows in an edible way.

§ 138. [Meanwhile, the establishment of a legal relationship between the greatness of the extremes and the number of values under which the extremes occur conflicts with a conception, for example, of DOVE and ENCKE, which, as a consequence, would elude the extremes of any legalism.]

DOVE, having stated in its first treatise [1)]: "On the non-periodic changes of temperature distribution on the surface of the earth", the extreme deviations of monthly and annual temperature means during a given number Years have taken place at various observatories, remarking explicitly: "the numbers given here are still somewhat arbitrary, since a single unusually severe winter or a very hot summer may perhaps double the differences found from a long series of previous years", a remark that is also schmid in his great meteorological works [2)]followed. Noticed Likewise Encke in his treatise on the method of least squares [3)] due to the fact that the extreme observation errors turn out in the known Bessel error rows a little too big against the theoretical requirement: "Incidentally, this difference is easily explained from the fact that larger errors usually require a very unusual combination of adverse effects, and are themselves often brought about by an isolated event, so that no theory can subject them to the bill. "

[1] Treatises of the Kgl. Berlin Academy of Sciences, from 1848.
[2] textbook of meteorology. Leipzig 1860.
[3] Berlin astronomer. Yearbook for 1834. p.249 flgd.

Accordingly, there has hitherto been no theoretical or empirical investigation and determination of the legal conditions of these values, and thus not only a certain gap in this respect should be filled by the following inquiry, but also the de facto elimination of suspicion. that the extreme values are not governed by law at all, take some interest in themselves.

It is true, however, that sometimes extreme or extreme deviations may arise from exceptional causes arising from the series of conditions under which a K.-G. considered as consisting and subject to investigation; z. B. barrel-shaped or decidedly microcephalous skull, where it is healthy skull. Such extremes are indeed unpredictable. But since the laws to be established only apply to such K.-G. If the propensities given earlier (chapter IV) are sufficient, the emergence of the extremes from the legal relations can almost be seen as an indication that these extremes are abnormal, which, in the case of normal circumstances, are to be excluded.

§ 139. Empirically, one can easily convince oneself of the change of extremes with the size of m in the following way.

Determine from the totality of a given list of given m , in which the measures are contained in random order, the two extremes E' and $E_,$ then, without changing the accidental order of the measures, divide the totality of them into a number of equal fractions. For example, if the total m = 1000 would be in 10 fractions of m = 100, determine the extremes of these fractions. If what not coincidentally, with a large total - m only exceptionally the case may be, happen the same extremes several times already in the totality, you will not find in the fractions, but they are on average only minor $E'$and greater $E_,$ give; and if the process is repeated on every fraction of m = 100, for example, by dividing it into 10 fractions of m = 10, the corresponding success will of course occur. Now one can regard the totality of the measures of given m, which one had first before oneself, as the fraction of a totality of larger m , and conclude that if one had several such fractions of the same m *in* front of them, the E' and $E_,$ obtained from them, also on average of the E' and $E_,$ the greater totality of all copies in plus and minus would be outbid.

It may be noted that the E , which are obtained from the equinumerous fractions thereof totality, have a slightly different size, and by even a fraction the totality than in other equinumerous fractions of a larger totality of given m can view, one would still between the e these larger fractions find differences, so that it therefore can not count on one of given $m$dependent very specific e' and $e_,$ to find; but in the first place it can certainly be said that normally in the sense introduced above, the E dependent on the given $m$on average, the higher the number of m , the higher the number of m ; secondly, for given m , consider their variation as a matter of uncertainty due to unbalanced contingencies pertaining to a closer examination, to return to below.

Let us explain the above with the student's chart [4)] with m = 2047, whose elements are given in § 65, according to which A_1 of the primary panel = 71.77; D_p after reduction to i = 1 inch but an average of 4 layers = 71.96. However, since the use of the whole m = 2047 would be enormously cumbersome, I use only 360 values as follows.

[4)] Because of the disadvantage of the non-uniform estimate, to which the measures of the recruits are at all subject, I would have preferred to have chosen another example, if original lists of other objects with equally sure pure randomness had been at the disposal of the measures; but this disadvantage can disproportionately disadvantage the conditions to which it depends in the future.

From the original list, in which the measurements are quite random, the first 18 measures of each of the 20 years were written out in their random sequence and combined into a totality of 360 measures. Here E' = 77.5, $E_,$ = 64 inches was found. Here Next these 360 measurements were in 180 fractions having a m divided = 2, in each of which of course, directly a measure that E', which other than $E_,$ occurs, and by dividing the sum of the thus obtained E' and $E_,$ with 180 were the mean E' = 73.16 and mean $E_,$ = 70.26 received; Furthermore, a division of the 360 measures was carried out in 120 fractions with a m = 3, whose mean E' and $E_,$ calculated and so forth, of which the results are summarized in the following table.

I. Mean values of the upper and lower extremes of *n* fractions with *m* members each .

m	n	e'	$E_,$	$E'-E_,$	$E'+E_,$
2	180	73.16	70.26	2.90	143.42
3	120	73.81	69.56	4.25	143.37
4	90	74.25	69.17	5.08	143.42
6	60	74.68	68.41	6.27	143.09
9	40	75.09	67.86	7.23	142.95
18	20	75.84	66.85	8.99	142.69
36	10	76.25	66.27	9.98	142.52
72	5	76,90	65.70	11,20	142.60
360	1	77,50	64,00	13.50	141.50

This table gives rise to the following remarks.

Without exception, one sees with increasing m the mean E', which decrease $E_,$ of which the natural consequence is that the difference between the two extremes $E'-E_,$ growing with increasing m, is nothing less than proportional with, as we can see m grows by z. B. at m = 2 equal to 2.9, at m = 360 is equal to 13.5. It may at first seem conspicuous that the sum of both extremes changes only very insignificantly with increasing m; that is, apart from the small irregularities at m= 4 and 72, which consider as the cause of unbalanced contingencies, the change in a continuous decrease of $E' + E_,$ with increasing m. But it is to be understood that way. Of course, when E' grows with increasing m, $E_,$ decreases, there is generally the possibility that both are just compensated, where then $E' + E_,$ with increasing $m$to remain constant, a case which, apart from unbalanced contingencies, would be expected if symmetry of deviations on both sides of the arithmetic mean existed. Now the measures of the recruits of one approach, but as they do not quite correspond to them, the result for $E + E_,$ too, does not quite correspond to the presupposition of such.

§ 140. [Although the values of Table I above clearly show the growth of the upper extremes and the decrease of the lower ones for growing m, they do not lend themselves to the proving of the extreme laws to be established in the following (§ 141). For these are to be deduced from the GG, which refers to the deviations from the arithmetic mean A or from the densest values D, so that the extreme determinations concern directly the extreme deviations from the initial values and not the extreme values E' and $E_,$ directly. The difference in the mode of determination which results from this is evident from the observation that $E'E$ may well be below the initial value and vice versa $E_,$ above it, and that then the deviation of that extreme from the initial value does not represent both the maximum value and rather the minimum value of the occurring deviations. The average values in the above table can therefore not be regarded as average values of the extreme deviations, since as such only the maxima of the deviation values are to be taken into account. However, this objection raises the objection that the extremes E' and $E_,$ as such, without regard to the principal value chosen as the initial value, attract interest and require the establishment of directly valid laws; but this can only be done through the mediation of the laws which are valid for the extreme deviations, since the distribution law on which this is based refers to deviation values. It is therefore also first of all empirically proven the theoretical provisions for the extreme deviations.]

[For this purpose, the dimensions of the original list must be replaced by their deviations from the initial value while maintaining the existing order. If the latter is the arithmetic mean$A$, then the deviations Δ τακε τηε place of the a, either with or without divorce of the positive of the negative deviation values, according to which the GG only on the upper resp. lower deviations alone or on both together. When outputs of D, however, are the differences ∂' and $\partial_,$ instead of a setting, while the positive ∂' of the negative$\partial_,$ but because the two-sided G.G., which is now used, in principle demands the separation of the upper and the lower deviations and relates to both in different ways.]

[In the present case, considering the weak degree of asymmetry inherent in the recruiting measures, one can choose the arithmetic mean as the starting value, and considering the small total number of 360 measure values available, the positive and negative deviation values should not be separated be treated. Accordingly, I replace the 360 recruits by keeping their order by their deviations from *A*, which for the sake of simplicity was equal to 71.75 instead of exactly 71.77. Then the totality of the deviations contains an extreme deviation of the value 7.75, and each subdivision thereof likewise has one and only one extreme deviation value which, although originating either positive or negative, appears as an absolute value the deviations are only considered in their absolute values. Now the series of 360 deviations, just as above, the series of 360 measures, even in *n* fractions, each consisting of *m* values, decomposed and each time the general with *U*Note the extreme deviation to be noted, the following table is given, in which it is stated how often a deviation of a certain size among the *n* fractions occurred as an extreme deviation *U* ; Of course, for *m* = 1 the deviations themselves are at the same time taken as extreme deviations:

II. Counts how many times the extreme deviation *U* in *n* fractions, each with *m* occurred members.

U	*m* = 1 *n* = 360	*m* = 2 *n* = 180	*m* = 3 *n* = 120	*m* = 4 *n* = 90	*m* =6 *n* = 60	*m* = 9 *n* = 40	*m* = 18 *n* = 20	*m* = 36 *n* = 10	*m* = 72 *n* = 5	*m* = 360 *n* = 1
0.00	12	1								
0.25	28	1								
0.50	25	4								
0.75	21	9	1							
1.00	16	6	-	1						
1.25	31	11	4	-						
1.50	35	14	7	-						
1.75	29	13	5	2						
2.00	24	18	13	13	4	3				
2.25	23	12	9	5	2	-				
2.50	15	7	6	3	2	1				
2.75	16	9	7	4	1	-				
3.00	11	10	7	7	3	-				
3.25	12	8th	7	5	3	1				
3.50	5	4	4	4	3	3				
3.75	16	14	11	9	8th	5	1			
4.00	7	5	6	5	4	2	1			

4.25	10	10	10	9	8th	6	3			
4.50	4	4	3	3	3	3	1			
4.75	3	3	3	3	3	2	2			
5.00	5	5	5	5	5	4	2	2		
5.25	6	6	6	6	5	4	4	3	2	
5.50	1	1	1	1	1	1	1	-	-	
5.75	2	2	2	2	2	2	2	2	-	
6.00	1	1	1	1	1	1	1	1	1	
6.25	-	-	-	-	-	-	-	-	-	
6.50	1	1	1	1	1	1	1	1	1	
6.75	-	-	-	-	-	-	-	-	-	
7.00	-	-	-	-	-	-	-	-	-	
7.25	-	-	-	-	-	-	-	-	-	
7.50	-	-	-	-	-	-	-	-	-	
7.75	1	1	1	1	1	1	1	1	1	1

This series, which represent distribution panels for extreme deviations can already by the successive advance of the smallest values the growth of the Extreme with increasing *m*recognize. However, a more precise idea thereof provides the following set of average values of the *U,* as which the arithmetic mean U_a, the central value U_c and the densest value $U_d$are intended:

III. The mean values U_a, U_c and U_d of the extreme deviations from *m*- shaped fractions.

	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 6$	$m = 9$	$m = 18$	$m = 36$	$m = 72$	$m = 360$
U_a	2.00	2.72	3.27	3.61	4.10	4.39	5.14	5.75	6.15	7.75
U_c	1.73	2.41	3.16	3.65	4.13	4.33	5.13	5.50	6.00	7.75
U_d	1.50	2.00	2.00	2.00	4.00	4.25	5.25	5.25	5.25	7.75

It should be noted that U_c *was* determined by simple interpolation, but U_d as the value to which the largest number of *U* fell; only for *m* = 6 the mean of the two values was taken, which together have the maximum number 8. Apart from the uncertainmost dense values, these values show a constant increase with increasing m . But U_d does not decrease, but retains its value only twice for every three consecutive m .]

[If one had separated the upper from the lower deviations to combine place both in a row, so would be entered II two tables in place of a table, one for the Δ ', the other for the Δ , ; however, since the total number of deviations for each would have been reduced by about half, the uncertainty of the provisions would have become considerably greater. If *D had been*chosen as the initial value instead of *A* , then a separation of the series of deviation values into a series of ∂ ' and one of ∂ , in principle, would have to be demanded.]

§ 141. [In order to set aside theoretical determinations for these empirical values, the probability law *W* [*U*] is to be derived, which states with which W. under *m* deviation values the extreme value *U is* to be expected. But if *U is to represent* the extreme value, then one of the *m* deviations must have that value, while the *m* - 1 can assume any other values between 0 and *U.* The law *W* [*U*] thus expresses W. that, of *m* deviations, any one is equal to *U* and the others between the limits 0 and*U* hold.]

[It is now, when the absolute values of the deviations are denoted by Θ , that W., that a deviation between the infinitesimally close limits Θ and Θ + *d* Θ falls, is equal to:

, (1)

It does not matter whether at the exit from the arithmetic mean the mutual deviations + Δ and - Δ or at the exit from the densest values the unilateral deviations ∂ ' resp. ∂ , under which Θ αρε to be understood; if only in the first case *h* = 1: η , in the latter case *h* = 1: *e* ' resp. = 1: *e* , is set, where η ισ the mean value of Δ , *e* ' resp. *e* , the mean of the ∂ 'resp. ∂, represents. If, therefore, of the *m* deviations Θ_1 , Θ_2 ... $\Theta_{m,}$ for example, the first equal to *U* and each following be smaller or at most equal to *U* , then for those first the W .:

and for each following the W .:

,

The W. for the coincidence of *m* deviations, of which the first is equal to *U* , and each subsequent has an arbitrary value between 0 and *U* , is thus equal to:

However, this value determines W in the same way if, instead of the first deviation, one of the following is set equal to *U* , and each time the *m* - 1 others belong to the value ranges between 0 and *U.* Consequently, the W., that of *m* deviations is any one U, and the others between the bounds 0 and *U* hold or, in other words, the W., that *U is* the extreme value among *m* deviations, by:

where $t = hU$, (2)

shown. There

.

.

so you can also:

; ($t = hU$) (3)

put.]

[From the latter form of representation, it can be seen that the integral over $W[U]$ is directly determinable. This integral, taken between certain limits, expresses however the W. that the extreme deviation falls between those limits. It is therefore W. that the extreme deviation is less than $U_1 = t_1 : h$ and greater than $U_2 = t_2 : h$, equal to:

, (4)

so that in particular the W., that $U = t : h$ the upper resp. lower limit of extremes, by:

respectively.

referred to as.]

[If we now determine a value $U_c = t_c : h$ *of* the kind that

or , (5)

thus it is equally probable to obtain a larger or a smaller value than U_c when determining the extremes of *m* deviations . Accordingly, U_c *will represent* the central value or probable value for multiply-determined determination of the extreme deviation whose dependence on *m* indicates the formula (5) and whose numerical value for a given *m is found by* means of the *t*-table. The following compilation of the related *m* and t_c for some values of *m shows* the growth of this central value with increasing *m* to see.]

m	t_c		m	t_c		m	t_c
1	.4769		9	1.2628		500	2.2611
2	.7437		18	1.4689		1000	2.3988
3	.8936		36	1.6576		5000	2.6946
4	.9957		72	1.8319		10000	2.8134
6	1.1330		360	2.1933			

[Apart from the central values, it is of interest to know the value which has the largest W. as a single value. With sufficiently frequent repeated determination of the extreme, it manifests m deviations as the closest value and is theoretically determined as the maximum value of $W[U]$. It thus satisfies for $t = hU$ the equation:

.

or:

, (6)

and shall be denoted by $U_d = t_d : h$. The calculation of t_d from equation (6) for a given m is, like that of t_c, to be done by means of the t- table. One finds the following associated values of m and t_d:

m	t_d		m	t_d		m	t_d
1	0,000		9	1,194		500	2,203
2	0,620		18	1,404		1000	2,342
3	0.801		36	1,594		5000	2.641
4	0.914		72	1,770		10000	2,761
6	1,060		360	2,134			

These show that $t_d < t_c$, that is, U_d lies below U_c, but that as m *increases,* these values approach each other.]

[Finally, the arithmetic mean of the extreme deviations can also be determined. If one calls it U_a, one obtains from (2):

(7)

or - after partial integration -:

, (8th)

For m = 1, (7) $U_a = 1 : h$ di results in the simple mean of the deviations themselves. For m = 2 one obtains (8) U_a = : h , *ie* the mean of the deviations multiplied by = 1.4142 itself. For larger m , $\Phi[t]$ can be represented in series form according to § 118, and thus also U_a in a series can be developed. For example, one arrives in this way for m = 3:

or, there

.

to:

,

Thus, U_a becomes equal to the mean of the deviations multiplied by 1.6623.]

[Each of the three values U_c, U_d and U_a illustrates in a special way the dependency of the extreme deviations on the number m of deviations from which the determination is made. However, when comparing the theoretical values with the empirical ones, it is important to consider both the certainty of the empirical determination and the ease of theoretical calculation, and to consider with consideration which of the three values offers the greatest advantage. Now the calculation of the theoretical value of $U_{c\ is}$ more convenient than that of U_d or U_a, with respect to the empirical determination, however, $U_{d\ stands}$ behind U_c and U_a for safety, while U_c and U_a generally deserve equal confidence. It will therefore be with advantage of the central value U_c to compare the theory with the experience.]

[For the measures of the recruits for which the empirically determined values of $U_{c\ are\ listed}$ in Tab. III, this comparison leads to the following results, where the mean η oφ the simple deviations according to § 65 is set equal to 2.045, ie 1: $h = \eta$ = 3.625 is:

IV. Comparison of the theoretical values of U_c with the empirical fractions determined from *m*- containing fractions.

m	U_c		Diff.		m	U_c		Diff.
	theor.	empir.				theor.	empir.	

1	1.73	1.73	0		9	4.58	4.33	- 0.25
2	2.70	2.41	- 0,29		18	5.32	5.13	- 0.19
3	3.24	3.16	- 0.08		36	6.01	5.50	- 0,51
4	3.61	3.65	+0.04		72	6.64	6.00	- 0.64
6	4.11	4.13	+0.03		360	7.95	7.75	- 0.20

In particular, given the small number of 360 values subject to the empirical determination, the agreement of the theoretical and empirical values will undoubtedly be found satisfactory, so that the established law of probability is confirmed by experience.]

§ 142. [The most important implications of the above developments are these:

l) Is a K.-G. with essential asymmetry-as presupposed as a rule-and has the two-sided GG for the same validity, then if $t' = U' : e'$ is set, the W .:

(9)

that the extreme value of the ***m*'** above *D* lies equal to *U'* and hence the upper extreme itself:

(9a)

be. The W .:

(10)

that *U* , = *t* , *e* , the extreme value of ***m*** **,** deviations below *D* , or even the lower extreme itself

(10a)

be. Is it possible, in repeated repetition again and again ***m*'** above and ***m*** **,** below *D* located copies of this K.-G. to select at random, the central value of the resulting upper and lower extremes becomes:

; Where

; where (11)

the densest value by:

; Where

; where (12)

the arithmetic mean by:

; Where

; where (13)

be presented.]

[2) Since with increasing m' and $m,$ the belonging them according to the above formulas values t' and $t,$ grow, so initially possess the difference values $t' - t,$ and $m' - m'$ have the same sign; Further, since according to the proportional law also $e' - e,$ the same sign as $m' - m,$ has, as the same is true of the differences $E't' - e,$ $t,$ and $m' - m,$, The asymmetry of the extreme deviations *D* thus has the same direction as the asymmetry of the deviation numbers. *D.* If one wants to refer this law to the deviations. of the arithmetic mean*A* , one arrives at the reverse law specified secondarily in § 33 (7) on the basis of the following consideration. Since the extreme deviations are large and subject to relatively large fluctuations, the assumption is made that the difference of the deviations does not change their sign when one goes from *D* to the relatively close value *A.* The difference of the deviation numbers bez. *A*but has the opposite sign as the difference of the deviation numbers bez. D. Thus, if the assumption is true, it has the difference of extreme deviations. *A* the opposite sign as the difference between the deviation numbers bez. *A.* In fact, this inversion law finds z. In Tables III and IV of XXV. Chapter for the members of the rye straws (with only one exception among 15 different cases) his probation. However, the same can only be considered as an empirical law which, in the case of substantial asymmetry, usually holds. In the case of insignificant asymmetry, on the other hand, it would no longer be able to assert its validity (see § 181).]

[3) disappears the asymmetry of K.-G. so are to be required for the extreme differences in principle the same values as its output value now that with *D* coinciding *A* bipartite must apply using the services of the simple GG instead. In this case, the formulas given under 1) remain, if only m' and $m,$ by ½ m and e' and $e,$ by the η valid for both sides equally ηreplaced. But, for essential symmetry, the law of distribution on the basis of the total m on both sides of *A*It is more appropriate to subject the positive and negative deviations together to the extreme, which leads to the following statements. If one puts $t = U: \eta$, then the W .:

(14)

ensure that the extreme values of the deviations ± *Δ* c .. *A* equal to *U* is. However, it remains undecided whether *U should be added to* the initial value in a positive or negative sense. It can therefore only be said that either

or (14a)

is, and at the same time in the former case *E* , above *A* - *U,* in the latter case *E* ′ remains below *A* + *U.* Corresponding remarks are also to be made regarding the addition of the mean extreme deviation values U_c, U_d and U_a to be determined according to the formulas (5), (6) and (8) to the initial values. Because this does not give you the middle extremes themselves, but only an upper resp. lower limit for the upper resp. lower middle extreme.]

XXI. The logarithmic treatment of collective objects .

§ 143. [The arithmetic treatment of K.-G. has the premise that the measures have a small proportionate fluctuation around the main values. But there are also K.-G., such as the dimensions of the gallery paintings and the daily rain heights, which according to a remark of the IV. Chapter in relation to the main values offer a very strong mean deviation, thus avoiding the application of the arithmetic treatment, On the other hand, the logarithmic treatment is accessible and enables a thorough proof of the logarithmic law of distribution.]

[This raises the task of discussing the logarithmic treatment in more detail in addition to what was already said in Chapter V (§35 and 36). There the general points of view were developed, which make it appear imperative, the distribution law of the K.-G. In principle, we should refer to deviations in relation to ratios rather than arithmetical deviations, from which the immediate conclusion was that instead of the arithmetic $\Theta = a - H$ the logarithms of the ratio deviations $\psi = a : H$, namely $\log \psi = \log a - \log H$ to be based on. Also the application of the logarithmic treatment of the main thing was already communicated there and the designation manner fixed. Accordingly, in general:

$\alpha = \log a$; ; $\lambda' = \log \psi' =$

$\log a' - \log H$; ;

$\lambda_, = \log \psi_, = \log H -$

$\log a_,$ (1)

and, in particular , to denote the densest value of a by $\mathbf{D}$, its arithmetic mean by $\mathbf{G}$ and its central value by $\mathbf{C}$, while the upper and lower deviation numbers and mean deviations refer to.$\mathbf{D}$ in the same way as bez. D by $\mathbf{m'}$, $\mathbf{m}$, and e', e, are given, so that:

$$; \quad ; \text{ where } \lambda' = a' - \mathbf{D}\ ; \lambda = \mathbf{D} - a\ . \quad (2)$$

If one also wishes to move from the logarithmic values to the numerical values which belong to them according to the logarithmic tables, then so

$$\mathbf{D} = \log \mathbf{T}\ ; \mathbf{C} = \log C;\ \mathbf{G} = \log G \ (3)$$

presuppose. It then designates $\mathbf{T}$ the closest ratio value of a , which is different from the arithmetically dense D value ; C agrees with the arithmetic mean; and G represents the geometric mean of aBy referring to these stipulations and developments of the specified chapter, however, the obligation, what was only there to promise, connects here. On the one hand, therefore, empirical evidence must be provided that the advantage of the logarithmic treatment for K.-G. distinctly pronounced with strong relative fluctuation. Another part of it is that for the logarithmic deviations of a and its main values $\mathbf{D}$, $\mathbf{C}$, $\mathbf{G}$, due to the two-column G. G . directly applicable provisions on the relative deviations of a and their principal values $\mathbf{T}$, C ,G and to derive a relationship between the logarithmic and arithmetic treatment by deriving the theoretically valid relationship between $\mathbf{T}$ and D.]

[Here, the logarithmic law of distribution itself is to be regarded as a law of experience sufficient in the case of a strong fluctuation, which, in the case of weak fluctuation, passes into the ordinary arithmetical law. Therefore, just as much as this, from the empirical viewpoint, does not require further justification. But in addition to the XIX. Chapter a hypothesis as to the origin, the K-G, have been prepared, from which the two-sided G. G, approximate arithmetic discrepancies arose, it seems necessary to modify this hypothesis so that from it for logarithmic deviations the distribution law follows in a similar manner. This is to be done in addition to this chapter.]

§ 144. [To illustrate the advantage that the logarithmic treatment has over the arithmetic of large fluctuations, I take each of the above-mentioned K.-G., the dimensions of the gallery paintings and the daily rains, as an example and share the results for both treatments.]

[From the catalogs of the older Pinakothek to Munich and the collection of paintings to Darmstadt, the dimensions of 253 genre images were found, the height dimensions were placed in a primary distribution panel. The centimeter was chosen as the unit of measure. The smallest measure was equal to 13, the largest equal to 265, the arithmetic mean A_1 equal to 54.4 and the central value C_1 equal to 44.2 cm. From this, a reduced board was obtained, in which the dimensions were summarized for each 10 cm. This resulted in arithmetic treatment according to the bilateral law. G. on the following results:

I. Height dimension of genre pictures in arithmetic treatment.

$m = 253$; $i = 10$; $A_1 = 54.4$; $E = 1$ cm .

a	z	
	empir.	theor.
-	-	1
15	13	15
25	41	38
35	54	39 [1)]
45	43	36
55	22	31
65	20.5	26
75	15	21
85	10	16
95	8.5	11
105	5	8th
115	3	5
125	6	3
135	3	2
145	5	1
155	0	-
165	1	-
195	1	-
235	1	-
265	1	-

$A_2 = 55.3$

$C_2 = 44.3$

$D_i = 35.4$

$D_p = 24.9$

$m' = 220$

$m_, = 33$

$e' =$ 35.8

$e_, = 5.4$

[1] [Here the maximum of the theoretical values does not fall on the interval 20 - 30, which includes the densest value D_p . However, this is conditioned only by the above intervalwise summary of the z . In fact, for another summary, for example:

intervals	z
20-24	14.0
24 - 28	15.9
28 - 32	15.8

so that a small excess belongs to the interval 24-28 with the densest value 24.9.]

But if we replace the a values in the primary table with the logarithmic values $\alpha = \log a$, which now vary between the limits α = 1, 11 and α = 2.42, we choose a reduced interval of size 0.08 Thus, if this table of α is treated in exactly the same way as the previous table of a , the following results are obtained:

II. Height dimension of genre images in logarithmic treatment.

$i = 0.08$; $m = 253$.

a	*Z*	
	empir	theor.
1.04	-	0.5
1.12	4	1.5
1.20	5	4

1.28	5	10
1.36	19	18
1.44	22	27
1.52	38	32
1.60	32	32
1.68	31	30
1.76	26	26
1.84	18	22
1.92	19	17
2.00	13	12
2.08	9	8.5
2.16	8th	5.5
2.24	1	3
2.32	1	2
2.40	2	1
2.48	-	1

G = 1.699

C =1.644

D_i = 1.538

D_p =1.549

G= 46.7

C= 44.1

T_i = 34.5

T_p = 35.4

m'= 165

$m_,$ = 88

e' = 0.256

$e_,$ = 0.136

If one compares both tables, the advantage of the logarithmic treatment becomes clear. For in the arithmetic table the sum of the absolute differences between empirical and theoretical values is equal to 74; in the logarithmic table, on the other hand, it is only equal to 37, which is exactly half the size. Furthermore, the empirical and theoretical densities, D_i and D_p, differ by 10.5 units; while the comparable values T_i and T_p differ only by 0.9. It should also be mentioned that the arithmetically determined quotient

the value 0.64, the logarithmically determined quotient

represents the value 0.792, so that it falls completely outside the theoretical limits of p , di 0.785 and 0.845, while it comes very close to the values required by the π laws ¼ π = 0.785 within those limits. All this shows that indeed the arithmetic treatment fails here, whereas the logarithmic one proves itself. It should be noted that despite the small *m of* the empirical panel, the emphasized relationships for the dimensions of the genre images are to be considered typical.]

[As an example of the daily rains, rainfalls (molten snow or rain) that fell in Geneva during the years 1845 - 1892 in January are to be used in the meteorological tables of the Bibliothèque Universelle de Genève (Archives des Sciences Phys. under the heading "Eau tombée dans les 24 heures". The total number of rainy days during the designated period of 48 years is 477; for each of them the rain levels are given down to tenths of a millimeter. 16 rainy days are recorded with 0.0 mm; the biggest rainfall is 40,0; the arithmetic mean A_1 is 4.45; the central value C_1 is equal to 2.24 mm. The primary distribution panel became a reduced panel with the interval i = 1 mm, which gave the following values for arithmetic treatment:

III. The rain heights of the month January for Geneva in arithmetic treatment.

$m = 477$; $i = 1$; $A_1 = 4.45$; E = 1mm.

a	z	
	emp.	theor.
0.5	133	67
1.5	88	63
2.5	43.5	61
3.5	28	56
4.5	27	49
5.5	28	42
6.5	27.5	35
7.5	14.5	28
8.5	16	22

9.5	11.5	16
10.5	12	12
11.5	10	8th
12.5	6.5	6
13.5	5.5	4
14.5	3	2
15.5	3	2
16.5	2	1
17.5	5	1
18.5	1	-
19.5	3	-
20.5	0	-
21.5	3	-
22.5	0	-
23.5	2	-
28.5	1	-
30.5	1	-
32.5	1	-
40.0	1	-

$A_2 = 4.49$
$C_2 = 2.40$
$D_i = 0.75$

$D_p = 0$

$e' = A_2$

$e_, = 0$

$m' = m$

$m_, = 0$

As you can see, the daily rain heights make a K.-G. with infinitely large asymmetry, in which $D_p = 0$, and thus all values above D_p. But the theoretical values of the *z are* so little in accord with the empirical ones that the arithmetical treatment proves to be inapplicable. However, if one wishes to proceed to logarithmic treatment, an agreement must first be reached on the conception of the 16 days of rain, which are recorded at 0.0 mm, for on those days the rainfall was not completely zero, but only so small that she did not reach a tenth of a millimeter. Therefore I assume 0.05 mm instead of 0.0 mm, so that the logarithms of *a*between the limits - 1.30 and + 1.60 vary. If, on the basis of this basically arbitrary fixing, one reduces the primary table to an interval of size 0.2, and chooses the lower limit of the first interval - 1.50, the following results are obtained:

IV. The rainy seasons of the month of January for Geneva in logarithmic treatment.

$m = 477; i = 0.2.$

a	Z	
	empir.	theor.
-	-	5
- 1.4	8th	4
- 1,2	8th	6
- 1.0	9	9
- 0.8	9	14
- 0.6	28	19
- 0,4	14	26
- 0.2	34	34
0.0	45	42
+ 0.2	66	50
+ 0.4	47	56
+ 0.6	53	60
+ 0.8	67	63
+ 1.0	53	52
+ 1,2	27	27
+ 1.4	7	8th
+ 1.6	2	2

G = 0.313 *G* = 2.06

C = 0.374 *C* = 2.37

$D_i = 0.800$ $T_i = 6.31$

$D_p = 0.843$ $T_p = 6.97$

$e' = 0.219$

$e_, = 0.749$

$m' = 108$

$m_, = 369$

Although the drawings here located below the densest value of *z* at - 0.4 and + 0.2 strong irregularities that do not disappear when changing the reduction layer, but rather by the course of *z* in the primary table and the summary due to the logarithmic intervals are; nevertheless, the correspondence between theory and experience is so good that the differences between the theoretical values and the empirical represent themselves as an adjustment of the contingencies inherent in the latter. Thus, the logarithmic law of distribution also proves to be quite satisfactory at the rainy heights.]

§ 145. [On the basis of the comparison between theory and experience described above, the logarithmic distribution law for K.-G. with strong relative variation as true. Since, after the discussions of Chapter V, the same thing is remarkably consistent with a weak, relative fluctuation of the individual values around the principal values with the arithmetic generalization of the GG, it is, as at the end of Chap. already been emphasized - at all as the strictly valid distribution law of the K.-G. to claim something. Thus, the probability determined W' or $W_,$ that a deviation from the logarithmic densely values **D** between the infinite bounds λ 'and λ ' + d λ ' or $\lambda_,$ and $\lambda_, + d\lambda_,$ fall for each K.-G. by:

;

; (4)

and the number of deviations between the specified limits is the same:

$$z' = W' \cdot m'; \; z_{,} = W_{,} m_{,} \quad (5)$$

where $h'm' = h_{,} m_{,}$; $h' = 1 : e'$; $h_{,} = 1 : e_{,}$ and e', $e_{,}$, m', $m_{,}$ to D can be obtained as an output value].

[For the principal values G, C and D of the logarithmic deviations, therefore, the same laws apply, which in the XIX. Chapter for the arithmetic mean values A, C and D were derived. But if G, C, and D are successively replaced by log G, log C, and log T, one obtains directly the laws valid for the principal values G, C, and T of the ratio deviations.]

[This results in particular the following provisions:

1.the central value C always lies between the geometric mean value G and the densest ratio value T, since according to the position law the same applies to C, G and D.

2.If one describes the geometric mean of above resp. lying below T a values by G' resp. $G_{,}$, such that:

$$e' = \log G' - \log T; \; e_{,} = \log t - \log G_{,},$$

so is due to the proportional law:

$$e' - e_{,} = \log G - \log T; \quad (6)$$

$$G' \xi G_{,} = G \cdot T.$$

3.If, as in § 131 with respect to D, we determine the value t'' in relation to D as well:

where *m* ″the larger and *m* " , the smaller of the two deviation numbers m ′ and m , imagines, then:

log C - log T = t "e" ; (7)

the difference of the logarithms being taken into account only in absolute terms. In the case of weak asymmetry follows from this:

.

or with regard to (6):

log *C* - log *T* = (log *G* - log *T*), (8)

an equation containing the π laws for the ratio deviations.]

[Finally, the relationship between the arithmetic principal values and those of the ratio deviations is given by the following sentences.]

For logarithmic mean values *G* = ∑ log *a: m* taken as the logarithm includes the with *G* to be referred to, so-called geometric mean or ratio value which always recklessly to a certain distribution law slightly smaller than the arithmetic mean value *A* = ∑ *a: m* and (after a Proofs by SCHEIBNER [2]) has approximately the following relation to *A* , which applies the more precisely, the smaller the so-called quadratic mean error, denoted by *q* , is. *A* , di *q* = is:

, (9)

After this you can *G* of approximate *A* derived.

[2] [W . SCHEIBNER, About means. Excerpt from a letter addressed to Prof. FECHNER. Reports of the Kgl. Sächs. Gesellsch. d. Scientific. Mathematics and Phys. Class. 1873. p. 562 flgd.]

The relationship between the logarithmically closest value *D* and the logarithm of the arithmetically dense value *D* is as follows:

, (10)

Therein, *e* , the lower logarithmic mean deviation = ∑ λ , : *m* , Mod the modulus of our usual logarithmic system = 0.43429, π as always 3.14159. This relationship is linked to the validity of the logarithmic generalization of the GG and can therefore be used in the empirical validations of this generalization.

[**Proof** . The logarithmically denominated value $\boldsymbol{D}$ designates the logarithmic interval which combines most z of all intervals of the same size . It is, therefore, by the maximum of the probability function (4) at constant $d\lambda'$ and $d\lambda,$, di by the output value of the deviations λ 'and $\lambda,$ determined. The arithmetically closest value $D,$ on the other hand, lies in that arithmetic interval which, at all intervals of the same magnitude, is the maximum - z, owns. Therefore, if the logarithmic law of distribution is valid, this value is found to be the maximum of the probability function (4) related to constant arithmetic intervals. Accordingly, we denote the arithmetic deviations of a of the densest ratio values $\boldsymbol{T}$ by $\Theta' = a' - \boldsymbol{T}$ and $\Theta, = \boldsymbol{T} - A,$ so that $d\Theta' = da'$ and $d\Theta, = - da,$ and set on the basis of of the definitions $\lambda' = \log a' - \boldsymbol{D} = \log a' - \log \boldsymbol{T}$; $\lambda, = \boldsymbol{D} - \log a, = \log \boldsymbol{T} - \log a,$ in the functions (4):

, (11)

Then one obtains for constant $d\Theta'$ and $d\Theta,$ for the determination of the maximum of:

;

the equations:

;

,

But the λ' and λ αρε , by their very nature, positive. It therefore offers only the second of the two equations a maximum for:

(12)

. Substituting is here to be $\lambda,$ corresponding a value of D to denote:

$\lambda, = \boldsymbol{D} - \log D$; furthermore

,

in fact, one obtains the relationship represented by (10).]

Section 146. [**Addition** . In accordance with the statement in § 35, the principle is laid down that the size changes of the copies of a K.-G. are significantly dependent on the size of the specimens, which undergo the changes, the result is immediately the modification that at the addition to the XIX. Chapter (§ 136) developed hypothesis is to make them serve the logarithmic distribution law.]

[For the derivation of the logarithmic law as well as for the derivation of the arithmetical special influences or circumstances, forces can be presupposed as causes of the size changes. Their number is indefinitely large, to assume *n* , and to *assign* to them all in the same way W. p for their intervention, W. $q = 1 - p$ for the absence of their effect. The success of its occurrence is, however, no more than an additive addition, but to be understood as a multiplication, so that instead of $a + i$ and $a + xi$, ai and ai^x occurs. This yields due to this modification for a copy of the size ai^x the same W. that the earlier developed hypothesis is a specimen of the size of $a + xi$ approached, so that now:

, (13)

But if one sets $\alpha = \log a$ and $\boldsymbol{i} = \log i$, then $\alpha + x\boldsymbol{i} = \log (ai^x)$, and we obtain as expression for the W. that the logarithm of the size of a specimen is equal to $\alpha + x\boldsymbol{i}$:

, (14)

According to this, the earlier developments apply in the same way and in the same extent to the logarithmic distribution law, if only everywhere *a is* replaced by $\alpha = \log a$ and i by $i = \log \boldsymbol{i}$.]

XXII. Collective treatment of relationships between dimensions. Medium proportions.

§ 147. According to this, I shall say something of a task which plays a considerable part in collective measurement, and the discussion of which may expediently find a place here, since the need of a logarithmic treatment is immediately suggested by it.

It is noteworthy that not only simple dimensions of an object but also relations of the same can be treated collectively, and I have already mentioned in this regard the relations between the cranial dimensions of a given race and the stem divisions, so-called limbs or internodes of one Graminee, for which you can find enough other examples. Let us consider the relationship between the vertical dimension *a* and the corresponding horizontal *b of* the skull of a given race, which is to be compared with other races, and as a rule puts *a* into the counter, *b*in the denominator, although the ratio can just as well be reversed. The ratio $a : b$ is already somewhat different between the specimens of the same race; but for the comparative characteristic of other races, instead of the changeable individual determinations, there are uniform results. Therefore, one can only demand an average ratio between *b* and *a* , which is generally called *M* [*a* **:** *b*]. After considering the arithmetic or geometric mean, *A* or *G takes* the place of *M.*The corresponding object can be set up with regard to the dimensions of the same part or dimensions of different parts not only of the human but of any object. So one can ask, how on average does the length of one finger relate to the other, the length of one link to the length of the second link of an ear, the length to the width of a business card, the mean temperature of one month to that of another, etc in short, the same task is endless.

§ 148. A middle relationship can now be obtained in different ways; in particular to the following, whereby mutually corresponding values of *a* and *b* are to be designated with the same index. The examples established for the direction *a : b* can of course be implemented for the direction *b : a* .

1) The arithmetic mean of ratios *A* [*a* **:** *b*] is obtained by adding all the individual values *a* **:** *b* and dividing them by their number; so:

, (1)

2) By summative means I mean that which is obtained by dividing the sum of all *a* by the sum of all *b,* or, what comes to the same thing, the arithmetic mean of all *a* by the arithmetic mean of all *b* , according to the formula:

, (2)

It could be argued against the use of this remedy that it is rather a relation between means as a means of circumstances; but as it is one, it is at the same time the other in the wider concept of the mean we use here, provided that, according to a definite principle, it is between the individual values of *a : b,* and indeed, except very exceptional cases, near the other means falls.

3) Percentage. To obtain this mean, form the values *a* **:** (*a* + *b*) and *b* **:** (*a* + *b*) and divide the sum of one by the other according to the formula:

, (3)

4) The geometric mean, represented by the formula:

, (4)

is the geometric mean of the product of the individual relations *a : b,* or, equivalently, the geometric mean of the product of *a* , divided by that of *b* , and in a practical way becomes the number sought in the logarithmic tables (∑ log *a* - ∑ log *b*) **:** *m* received.

If we now ask for the choice between these various meanings, it is first of all to be noted in general, as well as in regard to the simple measurements, that insofar as it is only a question of the relations of a K.-G. which allows a comparison of the same with other objects, each of the cited means contributes to such a characteristic only from another point of view, and that where the ratio *a* **:** *b* varies only relatively little at all, all four modes of determination lead almost to the same value , So z. B. 10 business cards, randomly pulled out of a package, if the short side with *a,* which is long denoted by *b* , as a means:

arithmetic 0.5654

summarily 0.5634

percent 0.5650

geometric 0.5649.

The extreme values *a* **:** *b* were 0.5333 and 0.6053.

In the meantime, where the fluctuations between the *a* **:** *b are* significant, the various mean determinations may give a considerably different result, and in general it is necessary to indicate the points of view which may decide the choice of one mode of determination over the other.

In this respect, it can generally be said that the arithmetic and the means of proportion in every respect are inferior to the other two meanings, and generally speaking the geometric means may merit preference, but also the summary may find useful use.

In fact, the arithmetic mean of conditions suffers from the following disadvantages.

a) In order to be able to add the individual fractions *a* **:** *b* , one must first reduce each one to a decimal fraction, which is very tedious for many values *a* **:** *b* .

b) In itself it does not matter whether one wants to use the direct values *a : b* or the reciprocal values *b* **:** *a* for the purpose of intermediate education, in order to determine the average ratio of *a* and *b* ; and of course one should obtain a consistent result in both ways; but this method does not grant, as it turns out, if one reverses the means derived from the reciprocal values, whereby one obtains the so-called harmonic mean to that obtained from the direct values; both do not agree, in short *A* [*a* **:** *b*] is not equal to the harmonic mean 1 **:** *A* [*b* **:** *a*] . Be z. For example, to take a very simple example of only two ratios:

; ;

so is:

; ; ; ; .

10 / 16 but = 0.625, 6 / 10 = 0.600. If we take farther apart fractions than in our example, the difference between the direct and harmonic mean becomes even

greater. In such K.-G., where most of the values *a : b are* not very far removed from a mean value, it is generally very small, but not everywhere negligible, and the method because of the ambiguity of its results in any case, to reject in principle.

c) If one has to determine the mean relations between three values *a* , *b* , *c* , then three ratios *a : b* , *b* **:** *c* , *a* **:** *c are possible* with their reciprocal values, and one may wish for two of these ratios (cf. direct or reciprocal) to directly derive the third. But this does not make this method by z. B. *A* [*a* **:** *c*] can not be obtained by substituting *A* [*a : b*] with *A* [*b* **:** *c*] multiplied.

The percentage means this shares all the disadvantages of the arithmetic. But one sometimes finds both one and the other needed.

The summary and geometric mean, however, are free from all these disadvantages. If, however, one were to give special trust to the direct arithmetical and equal principle of harmonic but of direct direct means, one would only be able to hold to the arithmetical or geometric mean of the direct and harmonic mean. But since it was also free, instead of *a* **:** *b* , from *b* **:** *a*To assume that this is a direct relationship would not only leave an ambiguity behind, but would also raise the question of choosing the arithmetic mean, whether one should prefer the direct or the harmonic, that is, that ambiguity can not be lifted from this side either. However, according to a proof, which I owe to Prof. SCHEIBNER [1] , the geometric mean of given conditions in the case of K.-G. In the usual case, that the direct and harmonic arithmetical mean differ only little, they coincide with the arithmetic mean of the two, and this can easily be confirmed by self-made examples.

[1] [Comp. W. SCHEIBNER: "About means", reports of Kgl. Saxon Society of Sciences. 1873. p 564. - After the given local provisions, the geometric mean is

approximately equal to: ,

the harmonic mean is equal to: ,

if *A is* the arithmetic mean and *q is* the mean square error; from which the above sentence follows.]

§ 149. Finally, it is only a question of how far the summary or geometric means is preferable.

Above all, the summary means is recommended by the ease of its determination, since it requires only the summation of all *a* , and all *b,* and the division of the one sum by the other, while it holds for the extraction of the geometric mean, first all *a* and *b* to translate into logarithms. Both have the following principal difference in meaning.

Be a summary:

It is clear that if, for example, one copy were very large in comparison with the others in terms of its two components a' and b', then the mean ratio would depend appreciably on the ratio $a' : b'$, and then $a'' + a''' + \cdots$ against a' and $b'' + b''' + \cdots$ against b' disappear, and that even the larger specimens, according to their size, gain more influence over the remedy. This is quite in order, if one gives larger specimens more weight for averaging than smaller ones, which may very well be the case, and in any case nothing prevents in the summary mean, what this circumstance carries, so well a characteristic relation of the given K.-G. to see, as in any other middle relationship, which does not carry him, by characterizing the object only in a different sense.

On the other hand, it may of course also be with the intention of letting large and small specimens of equal importance contribute to the determination of means, eg. For example, the relationship between the horizontal and vertical dimensions should not be more important for larger heads than for smaller ones, and this probably more frequent intention corresponds to the geometric mean.

The advantage, derived from the arithmetic and the means of the proposition, that, if three are determined on the average by three ratios $a: b$, $b: c$, $a: c$, the means of the third is directly derived from it, the summative mean divides it by the geometric one after both has:

, (5)

On the other hand, the cumulative mean has the following advantage over the geometric one. Suppose you have in a multi-limbed item, z. B. cereal halves of the given kind, in particular the average ratio of its length to the total length of Halmes summarily determined for each member, so you need these ratios only for any two members to add, so as to have the average ratio of the compound of these two members to the total length which is not the case with the geometric method, as it is easily proved; which can be expressed briefly: the mean of the proportional meanings of the parts and the whole depends more rationally on the summary procedure than on the geometric one and on any other.

In addition, the following case should be considered. Let's sit with a K.-G. occur among other specimens for which one or the other of both values *a* or *b is* zero; as for example For example, in determining the mean ratio between the weights of the solid and soft parts of different animals, some solid parts may be quite absent. In this case, the geometric mean becomes unusable because, depending on the zero value in the numerator or denominator, the mean becomes zero or infinity. Then one can only keep to the summary means, if one does not want to put forward the principle that such cases do not coincide with those where *a* and *b*to retain finite values everywhere, to unite under the same means.

§ 150. Since, in any case, the present subject is determined in various ways by the summary and geometrical relation of the components *a* and *b,* which enter into its purpose, it will, generally speaking, be exhaustive to determine that both means are to be determined. which does not prevent, according to circumstances, from making use of one over the other [2)] . But it has the determination of both except the general contribution to the characteristic of a given K.-G., whose components *a* and *b* are, nor the advantage that belong to the ratio of both means not unimportant special characteristic provisions, namely the following:

(1) If the ratio of *a* to *b,* regardless of the absolute size of *a* and *b* , is the same for all specimens, that is for large specimens as large as for small specimens, the sum-marian mean is equal to the geometrical one.

2) If *a* always increases or decreases at the same time as *b* , but not generally in the same ratio, then the ratio *a : b may* increase or decrease with increasing magnitude of *a* and *b* ; the former is the case when the geometric mean of the *a* **:** *b is* smaller than the summation, the latter when it is greater.

3) If the relative fluctuation of the values *a* by their arithmetic mean *A* equals the relative fluctuation of the values *b* by their arithmetic mean *B* , then the geometric mean is equal to the sum of the total. As a measure of the relative fluctuation applies here bez. *A is* the simple or quadratic mean deviation of *A* divided by *A* , namely $\varepsilon\alpha : A$ or $q\ \alpha : A$, let us say *P for*short ; corresponding to $\varepsilon\ \beta : B$ or$q\ \beta : B$, *Q for* short , with respect to *B.*

4) According to the relative fluctuation of the values, understood in the previous sense, is stronger by *A* or by *B* , the geometric mean is smaller or larger than the summation.

5) From the combination of 1) and 2) with 3) and 4) it follows further that, depending on the relative variation by *A* equal to that of *B* , greater or smaller, the value *a* **:** *b is*independent of the absolute values the *a* and *b* is constant or increasing magnitude of *a* and *b* is increased or decreased [provided that any value of *a* **:** *b* is a regular behavior, and only between Konstanz, continuous increase and continuous decrease allows a decision].

[2)] As good as two or more K.-G. Of course, according to the ratio of their means *A* and *G* , they can also be compared by the ratio of their *C* and *D* , and these results are by no means generally proportional. but I will not go into general discussions about this. - For example, in 237 German men's skulls, the mean ratio (Hor.: Vertik.) Of the vertical circumference of the skull capsule to the horizontal circumference was summarily 1.2830; geometric 1.2827; central 1.2837.

According to this, one can draw direct conclusions from the relation of the geometric to the summary mean, without making any further calculation, as the ratio *a* **:** *b increases* or decreases everywhere (or predominantly) as the size of an object and hence its components *a* and *b* increases , and whether one or the other component *a* , *b* fluctuates in greater proportion around their arithmetic mean.

The following as proof of the above sentences. On the first, the summary and geometric means are:

and

facing each other. Now CAUCHY proves in his cours d'analyze p. 15 and 447 that

generally between $a' : b'$, $a'' : b''$, ... falls. Now *,* if $a' : b'$, $a'' : b''$, ... *are* all equal to *a: b* , then the likelihood will be equal to *a : b* , while no less the geometric mean will be for the case of equality between $a' : b'$, $a'' : b''$, ... reduced to *a : b* . However, as the equality between the individual values *a* **:** *b* ceases, too, in general terms, the equality between the two remedies, and it may well be that *a* **:** *b* increases and partly decreases with changes in the absolute magnitudes of *a* and *b* , for which case nothing general is established leaves. But suppose, *a* and *b*they increase or decrease with each other at the same time, but they do not happen everywhere in the same proportion, there is a general proof for the proposition 2, which I owe to Professor SCHEIBNER, but which is cumbersome and not elementary, hence I here prefers to refer to the empirical proof of the rule by any self-made examples. And, of course, the rule will apply even if only *a* and *b* increase or decrease in the majority of cases with each other at the same time. Arriving at the third and fourth propositions, they are an inference of the relation between arithmetic and geometric mean of simple values given by SCHEIBNER[3] . After that you have to set *P* and*Q* as $q\ \alpha : A$ and $q\ \beta : B$:

;

; (6)

from which sentences 3) and 4) follow. If the formulas in question are only approximate, then the direction of the results is not changed by the omitted small links. The sentence 5) follows from the previous ones.

3) ["About means" aa O.]

§ 151. In the mode of determination of *G* [*a : b*] given above (§ 148) , the application of the logarithms merely serves to facilitate the calculation; but the need for their application reaches deeper.

The question arises whether, as well as the individual dimensions *a* and *b* , their relations *a* **:** *b* conform to our laws of distribution; a study in which, however, the decline to the individual *a : b* can not be spared, but from the outset it is evident from the remarks made hitherto that one can expect nothing from an arithmetical treatment of them; against which was the prospect that after finding the closest value of the log (*a* **:** *b*) the deviations of the individual log (*a* **:** *b*) of which could conform to our distribution laws, which can be found in the K.-G. confirmed found.

[To illustrate this by an example, I choose the ratio of the horizontal circumference to the vertical circumference (exact vertex) of the 500 European men's skulls provided to me by Prof. WELCKER. Since the horizontal circumference is consistently larger than the vertical - the smallest horizontal circumference (for a Little Russian) is 465 mm; the largest vertex arc (for a skull from the area around Halle) is 448 mm - so the ratios of all false fractions and their logarithms are positive. The minimum of the ratios is equal to 1.211, the maximum equal to 1.403. The logarithmic values thus vary between the limits 0.083 and 0.147; they have the mean $\boldsymbol{G}_1 = 0.1073$, so that the geometric mean G_1 the ratio is equal to 1.280. If one chooses the logarithmic interval i = 0.003 and the lower limit of the first interval the value 0.0825, one obtains the following comparison table between the empirical values and the theoretical values required by the logarithmic distribution law:

Ratio of the horizontal circumference *a* to the vertical circumference *b* for 500 European male skulls.

$\alpha = log\ a$ - log b ; i = 0.003; m = 500; $\boldsymbol{G}_1 = 0.1073$; $G_1 = 1.280$.

α	z	
	empir.	theor.
-	-	1
0.084	1	2
0.087	4	5
0,090	12	10
0.093	17	19
0.096	29	32
0,099	47	46
0,102	64	58.5
0.105	64	65
0.108	67	64
0,111	61	58
0.114	45	47
0,117	36	36

0,120	28	24.5
0.123	11	15
0.126	7	9
0,129	3	4.5
0.132	2	3
0.135	1	0.5
0.138	0	-
0.141	0	-
0.144	0	-
0,147	1	-
total	500	500

$$\mathbf{G}_2 = 0.1073 \quad G_2 = 1.280$$

$$\mathbf{C} = 0.1070 \quad C = 1.279$$

$$\mathbf{D}_i = 0.1068 \quad \boldsymbol{T}_i = 1.279$$

$$\mathbf{D}_p = 0.1060 \quad \boldsymbol{T}_p = 1.276$$

$$e' = 0.0079$$

$$e_, = 0.0066$$

$$\mathbf{m}' = 272.5$$

$$\mathbf{m}_, = 227.5$$

$$h' = 7142$$

$$h_, = 85.48.$$

It should be noted that $\mathbf{D}_i$ does not represent the empirically dense value directly deductible from the above table (which is rather equal to 0.1075) but the average of the three values calculated from the three possible reduction positions: 0.1075; 0,1085; 0,1043. This mode of determination was chosen because it happens that the reduction position is of great influence on the position of $\mathbf{D}_i$, while $\mathbf{G}_2$ and $\mathbf{C}$ almost completely coincide with the values resulting from the primary panel. The asymmetry is weak; as well as

close to ¼ $\pi = 0.785$. However, the correspondence between the empirical and theoretical z values is undoubtedly satisfactory.]

XXIII. dependencies

§ 152. One may ask whether the mean temperatures of the successive years vary according to pure laws of chance, or show a certain dependence in their succession on each other; a question that can be transferred to many analogous cases. Now the relationships of dependency may be different, and the investigations thereon accordingly different. One of the simplest questions and investigation ways but follows the following remark.

I take a list of drawn lottery numbers. One such example starts with:

26 826 _

21 460 +

31 094 _

22 120 _

16,226 (+)

Suppose, as applicable, every decrease from one to the next number with -, every increase with + and thus obtains without resorting to the first number the following series: - + - - and of this without resorting to the first sign two sign changes and a sequence of the same Character; or if I fall back both with number as sign: - + - - + and here four changes and a consequence; in general, if I denote the number of numbers *m* and the number of changes and sequences *z* , in the first case $z = m - 2$, in the latter case $z = m$. First hot method *a, the*latter method *b.*

If I now apply the method *a* or *b* , then if *m is* large, then I find the number of symbol changes to be approximately equal to twice the number of character sequences, so that I can assume the W. of one to W. the other as 2 **:** 1 [1)], This is the law of pure chance.

1) [Theoretically, this relationship is derived from the observation that three values *a, b* , *c,* which are free from the dependence on succession, are equally probable in each of the six successions:

a , *b, c* ,
c , *b, a* ,
b , *a, c,*
c, a, b,
a, c, b,

b, c, a

can occur, so that when z. B. $a < b < c$, the first two successions depending on a string, the last four each give a mark changes, and thus the W. a string equal to $1\,/_3$ the W. a Zei-chenwechsels equal $/\,2_3$ is to be set.]

But should there be a dependence of the successive numbers of the kind, that they rise and fall continuously in a certain interval, the number of strings would increase beyond the former ratio. Yes, if the dependence always proceeded in the same direction, then one would obtain by method *a of* all strings, by method *bm* - 2 sequences, 2 alternations.

Let's stick to method *a* stand and tell the number of exchange *w* that the consequences *for* , the full independence is $f = {}^1/_3$ *, for* the full dependence by *z f* = and partial dependence by values of *f* characterizes between these and one will find a measure of partial dependence given *f* and *z* in the ratio in which the excess of *f* exceeds the measure of full independence to the total excess of full dependence over full independence, that is, if we compare this measure with Abh . describe:

Dep. = . (1)

Meanwhile, *f* is uncertain because of the finite *m* , and Abh. Is involved in this uncertainty. The determination of this uncertainty is included in the value of Abh. As a probable error.

[This determination is made by calculating the probable bounds which result from the reversal of the so-called BERNOULLI's theorem for the W. of a string on the basis of the observed values of *f* and *z* . If one sets the unknown W. for the occurrence of a string equal to *x* , the W. of a change of sign equal to 1 *x* , the theorem of the theory of probability [2)]follows :

(2)

for the value of *x* between the boundaries:

and (2a)

liege. Since for $W = ½$ the value of $c = 0.476$ becomes 94, the probable limits of *x are* equal to:

, (3)

Accordingly, the probable limits of Abh. Are the same:

, (4)

It is thus to bet 1 to 1 that the degree of dependency as defined above is not less than the lower and not greater than the upper of the two specified limits.]

2) [Comp. MEYER's Lectures on Probability Chapter VII.]

[The same can also accept negative values and thus indicate a dependency that manifests itself by predominantly - in the extreme case by constant - change of the signs. This requires that the number *f* of the strings below the value of $^1/_3$ *for* decreasing and in the limiting case would equal 0th]

§ 153. [The application of the dependency measure (4) to check the succession dependence of meteorological monthly and daily values leads to the following results.]

[In one of his essays [3], DOVE compiles the "deviations of the individual months from the long-term mean values of the same" for a number of places. For Berlin, this compilation covers the period from 1719 to 1849 with the fall of only 3 to 7 years for the individual months. From this, for every month taken together, according to method *a,* 1421 successions of characters result, namely 913 character changes and 508 character sequences. The W. *x of* a string thus has the probable limits:

or 0.3575 ± 0.0086;

from which one

$$\text{Dep.} = 0.036 \pm 0.013$$

receives.]

3) [Report on the observations made in the years 1848 and 1849 at the stations of the meteorological institute. Berlin 1851. p. XX flgd.]

[In the Dutch Yearbook for Meteorology [4]] tables of daily thermometer and barometric deviations are found from the daily normal levels found from many years of observation, for the individual months of the year. The observatories are the various meteorological stations of the country; the observation times are specific hours of the day to which both the normal and the deviation values refer. This takes into account the legitimate rise or fall of the thermometer and barometer within a month, so that the succession dependence is not affected. I chose the values given for Utrecht in the month of January during the 10-year period from 1884 to 1893, at noon 2 o'clock. The same resulted in method *a* 298 successions of characters. Among them were 129 character strings and 169 character changes for the thermometer deviations, for the barometric deviations 153 character strings and 145 character changes. Hence, for the former, we find the probable limits of W. of a string:

$$0.433 \pm 0.019$$

and:

$$\text{Dep.} = 0.149 \pm 0.029;$$

for the latter, on the other hand, as probable limits of the W. of a string:

$$0.513 \pm 0.020 \text{ and:}$$

$$\text{Dep.} = 0.270 \pm 0.029.$$

Accordingly, the daily thermometer and barometric deviations have a decided succession dependence, while the same appears for the monthly temperature deviations - as already noted in § 20 - with little decisiveness.]

4) [Meteorological Jaarboek, uitgegeven door het Kon. Nederlandsch Meteorological Instituut. "Thermoen Barometer afwijkingen".]

[The daily rain heights, on the other hand, are - according to a remark in § 21 - free from essential succession dependence. In fact, in the XXI. Chapter as example of the logarithmic treatment of selected rain heights of the month January for Geneva from 1845-1892 under 475 successions of signs 165 sequences of the same signs. All the 477 values in their temporal succession are united in a row, and the successions of the same values have been taken into account, alternately, the increases and the decreases. Thus we find:

$$\text{Dep.} = 0.022 \pm 0.022.$$

From this value does not differ significantly the degree of dependency for the original list of the recruiting measures whose succession dependence is to be regarded as immaterial from the outset, since it is not clear how in the recruitment of the recruitment business a significant dependence in the order of the measures should arise. For the series of 360 student recruitment measures, which are described in chap. XX serve to prove the extreme laws, namely result in 125 character strings and 233 character changes, according to which

$$\text{Dep.} = 0.023 \pm 0.025$$

becomes. In both cases, the limits of the dependency measure include the value 0 of the ideal case of independence.]

§ 154. [Another way of investigating succession dependence was described in § 20 at the same time as previously discussed. It is based on the remark that, given full independence and without interference from unbalanced contingencies, the number of sequences of every two or two values below the middle of the scale C *is* equal to the number of alternations between every two above and below C values. Namely, the values above *C* +, the values below *C*by - denotes, the W. of a positive value is just as large as the W. of a negative; therefore, even with complete independence, each of the four possible successions is: + +; - -; + -; - + equal probably. However, the first two each result in a string, the last two each a character change, so that for both a string and for a character change the W. ½ exists. If, for a series of values treated in this way, f strings and w symbol changes are found for a sufficiently large number of $z = f + w$ successions of characters, the probable bounds for the unknown W. x *of* a string from the Reversal of BERNOULLI's theorem:

being found. Here, the value f z when taking place partial succession dependency that can be recognized as an accumulation of effects in comparison with the change between the values ½, which applies to full independence, and the values 1, that for $f = z$ full Indicates dependency. Again, in the ratio of the surplus of partial dependence over full independence, ie of the calculated x over ½, to the total excess of full dependence over full independence, that is, of 1 over ½, one can obtain a measure of dependency, and

Dep. = , (5)

or, if the probable limits are taken for x ,

Dep.

= (6)

put. This measure of dependence, too, retains its meaning for negative values, in that it then indicates the preponderance of the W. of a change of sign over the W. of a string.]

[As an example of this dependency determination serve partly the series of monthly deviations for Berlin, partly the series of the recruiting measures whose succession dependencies according to formula (4) have already been calculated, so that at the same time a comparison between both ways of determination becomes possible.]

[Regarding the monthly deviations, the value center is *C* for each monthto determine. It falls below it for a few months, for the majority of months above the respective long-term average. However, which greatly facilitates the application of this method, the mean itself may be taken as the center of value, so that the positive and negative values of deviation may at the same time be considered + values and values in the sense of our method. For the 12 months, taken together, yield 768 character strings and 665 character changes after determining the central values; however, with direct reference to the mean, there are 769 character strings and 664 character changes, which does not make a significant difference to the measure of dependency. From the former determinations, the probable limits for the W. of a string are the values:

$$0.536 \pm 0.009;$$

from the latter the values:

$$0.537 \pm 0.009;$$

and in the former case:

$$\text{Dep.} = 0.072 \pm 0.018$$

in the latter case:

$$\text{Dep.} = 0.073 \pm 0.018.$$

The dependency measure (6) thus leads to larger values than the dependency measure (4).]

[The central value *C of* the 360 recruits measures 71.75. According to this, there are 165 string sequences and 194 character changes among 359 successions of characters. The probable limits for the W. of an order are therefore:

$$0.460 \pm 0.018$$

and:

$$\text{Dep.} = -\ 0.081 \pm 0.035.$$

Accordingly, a relatively smaller value is obtained in this case than according to formula (4); however, it deviates to a greater extent from the ideal value of 0.]

§ 155. [The dependency measure (6) can also be used to determine the interdependent dependence of two dimensions of a multi-dimensional K.-G. or dimensions of different but temporally related K.-G. be made serviceable. For this purpose denote the growth of each of the two dimensions compared by +, the decrease by -, so that a series of *m* pairs of related values is characterized by *m* - 1 character pairs ++, - -, + -, - +. Under the latter, as long as the two dimensions are completely independent of each other and without the addition of unbalanced coincidences, there will be as many character strings as sign changes, since the W. is to be assumed to be the same for each of the four types of character pairs. It is therefore, if under *z* Observations *f* sequences and *w* alternations occur to calculate the W. of a string according to formula (3) and to determine the dependency measure according to formula (6).]

For example, between the size of the horizontal perimeter and the vertical vertex, the European male skull, which served as an example in the previous chapter for the treatment of relations between dimensions, has a dependency which can be determined as follows by the given method. The 500 skull masses are summarized in the original list in 34 groups of 6 to 30 skulls (the first two contain 20 Breisgau and 15 Swabians, the last two 6 Serbs and 22 Greater Russians); in each group, however, the measures are ordered by increasing horizontal extent. I now counted for each group the number of strings and character changes that result in the course of the two compared values, with the cases where a stoppage occurred in the change of either size, half of the consequences and half of the bills were added. After this there were 273 character strings and 193 character changes under 466 character pairs, so that:

Dep.

revealed.]

[A second example I take from Professor WELCKER in the treatise [5] : "the capacity and the three major diameters of the skull capsule" communicated dimensions of the interior *I*and the length *L,* width *B* and height *H* of 101 skulls of different peoples, in particular the dependence of WELCKER's "skull modulus" $L + B + H$ and the product $L \cdot B \cdot H$to calculate from the associated interior space. If the individual groups of skulls arranged according to increasing internal space, the number of which is 13, are treated here as well as given with regard to the groups of horizontal or vertical dimensions, then both $L + B + H$ and *I* and *L result. B. H* and *I* 59.5 strings versus 26.5 characters under 86 pairs of characters. It is therefore both for the dependence of the sum as the product of the three main diameters of the interior:

Dep. = ± 1.3490 = 0.384 ± 0.067

to put. Also, as Prof. WELCKER points out in the abovementioned treatise, it is possible to tabulate both the values of $L + B + H$ and those of $L \cdot B \cdot H$ average interior values which allow, on the basis of the measured value of the sum or of the product of the three major diameters to approximate the associated interior of the skull.]

[5] [Archive of Anthropology, Volume XVI, Issue 1 u. P. 72 flgd.]

[An intensification of this dependency determination is achieved if the size of the growth or the decrease is taken into account for the compared dimensions. This can be done by determining the weight of the observed strings and character changes in the following manner. Give a pair of characters the weight 1 as each dimension increases or decreases by the unit of measurement, and set the weight of each pair of signs equal to the product of the two quantities by which each of the two dimensions increases or decreases. In this way, instead of the last given dependency determination between the sum and the product of the three major diameters and the interior of the skull for $L + B + H$ and *I obtained* :

Dep. = 0.8436 ± 0.0012

for $L \cdot B \cdot H$ and *I* :

Dep. = 0.8387 ± 0.0008

in the first case, for *f* and *w,* the values 45641 and 3871; otherwise the values 99886 and 8763 occur. As might be expected, the degree of dependency has increased considerably without any substantial difference between the dependency ratios of $L + B + H$ and *I* and that of $L \cdot B \cdot H$ and *I*makes itself felt. Therefore, if, as

WELCKER's remarks show, the product of the three diameters provides a more sensitive interior space than their sum, it must be noted that our method, at least in the relatively small number of skulls, does not make such a distinction allowed. Since, moreover, this dependence determination is not influenced by the absolute size of the dimensions compared, but is based only on their increase and decrease, it can also give no numerical proof that - as WELCKER's treatise likewise teaches - the tabular assignment of interior values becomes significantly more accurate to the sum of the three major diameters when the so-called broad index of the skull, ie the ratio between its width and its length, Accordingly, the skulls of dolichocephalic, mesocephalic and brachycephalic form are treated separately. For this purpose, the ratios between the sum of the three diameters on the one hand and the interior space on the other hand would have to be subjected to collective treatment taking into account the breadth index.]

XXIV. On the spatial and temporal context of the variations in the size of the recruit.

§ 156. The crops not only produce a different crop depending on the nature of the vintages, but grow in different years up to a different height, which depends mainly on the temperature and humidity of the different vintages. Inasmuch as these conditions are common to larger stretches of land, their influence on the growth of crops in connection with all parts of such stretches is also asserted; But it changes from route to route as these conditions change.

The question arises as to whether there is something similar for the size of the people born in the same years, whether it also changes according to the nature of the vintages in a certain connection for contiguous tracts of land, perhaps even changes in connection with that of the plants. Of course, it is scarcely possible to presuppose a corresponding direct influence of temperature and humidity on the growth of man and of plants; Nor do men grow up from the germ any more than the crops in every year, nor do they complete their existence in the same year, so that one should only pay attention to the circumstances of a year; but it is conceivable that the fertility of a year, by influencing the nutritional relationships of the parents at the time of production of the child or during pregnancy, or of the child itself during the growing season, especially the first, also indirectly influenced the growth of the child, and in this respect really growth of plants and people changed in the context. But the nutritional conditions of men in a country do not depend only on the fertility of the years; the state of war and peace, the state of industry and commerce, influence it, and not only food relations can be considered; also everything relating to the physical and mental strength and health of the parents at the time of the generation of the child and during the pregnancy over a certain country in connection, maybe even epidemic and even cosmic influences. In short, one is not at all embarrassed to find possible causes that the average size of the man born in the same year changes as much as that of the plants, whether with or without relation, over larger distances in the context. The only question is whether the fact of such a connection can be traced over larger or smaller tracts of land; and the following investigation will prove that it is the case. Apart from this, the following study will deal with the question whether the influences which affect the change of size also betray a temporal connection of the kind, that instead of being irregular, in the sense of unbalanced contingencies, changing gradients and falling of the size measures over the vintages always several vintages are inclined to rise, and again several, to fall. For the twenty years of Saxon student recruits nothing of the kind can be proven, on the other hand results a more decided result for years of Belgian recruits.

In addition to the two previous questions, I have also examined the question of whether a relationship could be found between the principal fruit prices, which took place around the recuperation time of the recruits, and the average size of recruits from that time, and I have this investigation in RECLAM's hygienic journal "Health" (1876) [1)]; However, since it has led to a significantly negative result, I do not come back to it in the following.

[1)] [investigation on the spatial and temporal context in the diversity of human size; IV. Section: The question of how the size movement of the recruits is related to the movement of fruit prices around the time of birth. "Health", 1st year, S, 54 flgd.]

In any case, in order to examine the questions to be dealt with here, measures of the recruits combine several of the most favorable conditions; one would like to say that they are made to do so; are also the only material that is available for such an investigation so far. Once the recruiting measures of each year are taken by persons who were also born in the same year, 20, 19 or 18 years back, depending on the diversity of the countries. Secondly, the recruiting measures across all cultured countries extend through longer epochs, are specified by whole countries, regions, districts, cities, thus providing an opportunity to compare the effects of more general and more specific influences on a larger scale. Third, the number of individual measures, even for a moderate district,

For my part the whole investigation in relation to previous questions has been conducted only on the basis of the very limited material which I had in the Saxon and Belgian dimensions, which was partly due to the fact that I did not find other useful material, partly that investigation at all merely as a secondary investigation. Because for Saxony I could probably still url lists for other parts of the country and later vintages to procure; but even the processing of the previously used material was time and endowment. A more general examination of the questions dealt with here can only be a matter for statistical institutes which have sufficient mechanical computing power to work with an extensive material. which are in fact greatly claimed by such investigations. All the same, the following investigation, so far as it can be conducted, may retain the twofold interest, once it designates and discusses ways in which to carry out such an investigation, and secondly in the remarkable results which are thus given to limited spaces and epochs, contains an invitation for others to give the investigation further episode.

With these advantages, which the measures of the recruits could offer everywhere as a basis for investigations of this kind, it is only to be regretted, as has already been touched upon, that in the statistical works where the data were to be sought, they are in general inappropriate Form are offered. In some cases, *A's* annual mean values are not even found, in part not in sufficient extent or sequence, specialization, sharpness drawn, and the dimensional lists, as far as I know them, have nowhere been arranged so that they can be drawn with precision but require their drawing from primitive lists a laborious work, and the procurement of the original lists themselves is not available everywhere.

§ 157. Hereinafter the general description of the method of investigation.

Let's call the change of a size from one year to another movement of size at all and speak of a parallelism of the movement of two sizes, eg. For example, the annual mean of the measures of the recruits in two neighboring parts of the country, if the mutual movements have the same direction in decrease or increase, without requiring, as would be required in the mathematical meaning of the word parallelism, that the change of the two quantities being compared is also equal or go proportional to each other; enough if it only corresponds in the direction. A case of parallelism will be with | | , A case of Nichtparallelismus or, as we say anti-parallelism with э designated; the number of | | under a given number *z*compared motion cases with *p* , the э with *q* . If there is no dependence of both magnitudes on each other or on a common cause, the passage would be followed by a greater series of years, and thus of motion | with the э indifferent change, and the number of both close to each other, except for unbalanced contingencies, must be the same. If all cases fail in parallel, one would conclude that a cause or composition of several causes, which affects the movement of the two quantities, outweighs all those acting in the opposite sense. Should only be a significant overweight of | | about the эIn view of the greater preponderance, it would also be more likely to find that a common influence in this respect is taking place, but that there is sometimes room for a predominance of opposing influences. If finally the э exclusively or very predominantly occur, this would not prove an independence of both magnitudes from each other, but that the same influence, which works for the enlargement of the one size, works for the diminution of the other.

Apart from the parallelism and antiparallelism in the given sense, ignoring the size of the motions, this size can also be taken into account, as the dependence or common influence increases considerably, preferably when the motions are strong are those in which the parallelism or (in antagonistic action) anti-parallelism invariably or predominantly shows; whereas, in weaker motions, one must take into account the influence of unbalanced contingencies, and it is therefore appropriate in cases where there is a greater number of years (as in Tab. III, see §160), after first considering the movements after the series Vintages to see if the ratio of | | and эIn the course of time, it changes conspicuously to list them once again according to the order of magnitude of motion, one size or another, where the cases which are the precondition of common influence are then favored on the part of the larger, the unrelated, and the indifferently changing sides such minor influence must be acceptable.

This raises the question of whether the weight, what is a trap of | | or э has to be added to the sum or product of the quantities of movement involved. Indisputable to the product, because if one of the two movements entering into a case is zero, the weight of the case, as a tie between | | and э , must be zero, and because parallelism between positive movements is equal to that between negative movements, which can only be achieved by the product of both movements.

Having said this, one will have an even surer judgment than the sheer number of | | and э by the following consideration of the weights win. Take the motion products of related sizes for both the | | as э special, call the sum of the first P , that of the second Q , and judge now, instead of according to the relations or relative differences from p to q , from that of P to Q. If a common influence is to be acceptable, then not only must a significant relative preponderance of either one of the two values P , Qabove the other, but also the relative difference from p to q are exceeded, in short $(P - Q) : (P + Q)$ be greater than $(p - q) : (p + q)$ for absolute values , because in the latter circumstances the greater weight of the strong cases in favor of the influence does not come into consideration. It is therefore useful in any case, both p and q as P and $Q$to determine, if the inference to be drawn from the behavior of the first is not reinforced by the behavior of the second, to doubt the common influence.

The certainty of the conclusion grows on the one hand, on the one hand, with the number of cases of movement z , on the other hand, the size of the relative differences

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For far too small *for* or even low relative surpluses can be drawn no notable result at all; the more the two magnify each other, and the more strongly the second increases above the first, the nearer is the influence of certainty, and it would undoubtedly do nothing to prevent more precise determinations of probability in this respect, which I do not address here want [2)] .

[2)] [Comp. § 155. It is only necessary to interpret the parallelism as a string, the anti-parallelism as a change of sign, in order to gain a direct connection to the local provisions.]

§ 158. The movement of the measures may be followed on each of the principal values *A, C, D* , but the easiest provision gives the practical effect; and in this respect *C* is all the more in the advantage, as it is also obtainable from recruiting plates, in which after the so usual error for the number and suffix not also the preamble and the total is given. But if you want to save yourself the formation of a distribution panel completely, so the following procedure is recommended. Count the number of measures which are smaller, and those which are greater than a measure or a small measure interval determined once and for all, call the number of the first *k,* that of the other *g*and now judge by the parallelism or antiparallelism of the ratio $g : k$ or $g : m$. In the case of the Belgian measures I have assumed the interval 1618 to 1643 mm, where then *g* denotes the number of measures which are greater than the upper, and *k* the number of those which are smaller than the lower limit of this interval; and the following inquiry will show that the judgment is accordingly correct in the *C* judgment , in that I have used *G in* comparison with *C* for the Belgian measures $g : k$ and $g : m$. Since, however, I was able to obtain complete primary tables from the Saxon masses, from which precise arithmetic means A_{1} could be deduced, I have adhered to them.

Since the values A_1 , A_2 , C , $g : k$, $g : m$ do not change in exactly the same proportions, however, in the case of small *m* and weak motion, differences might occur according to the comparative course of the changes of one or other of these values; but for larger *m* and stronger motion which can only be a resounding result at all, parallelism, where one exists much is, can not be disturbed. This was for A_1 (primary), A_2 (reduced) and *C* (reduced) by comparison in this regard after the twenty years of the student recruitment board notice.

About the spatial relationship of the variations in the size of the recruit.

§ 159. In itself there is nothing remarkable in the fact that the average sizes of recruits vary in the same place; for who, in the multitude of accidental circumstances on which the growth of individuals depends, can expect that the differences in them will be balanced out by drawing on the same values one year as the other. However, it may seem conspicuous that the variations in the average size of the recruits between different years are large enough to be felt by recruits without the need for a recourse. So I was told at the district office in Leipzig, from which I obtained lists for the recruits from Leipzig, that we speak of good and bad years in this respect, and a senior Austrian officer, who for many years stood before the recruits, When he was told of my remarks made in this regard, he explained that there was no doubt that the size of the recruits changed by age. For I myself noticed when, in the light of my general investigation, I drew arithmetical means from the seventeen years of the Leipzig Stadtmaße, that the last year in 1862 gave the maximum, the penultimate of 1861 the minimum of all seventeen years, and the difference of 1.17 inches seemed to me so strange by its size that I tried to get to the bottom of it. From this the whole following investigation has taken the exit. When, in the light of my general investigation, I drew arithmetical meanings from the seventeen years of the Leipzig Stadtmaße, that the last year in 1862 gave the maximum, the penultimate of 1861 the minimum of all seventeen years, and the difference of 1.17 inches seemed so remarkable to me by its size. that I tried to get to the bottom of it. From this the whole following investigation has taken the exit. When, in the light of my general investigation, I drew arithmetical meanings from the seventeen years of the Leipzig Stadtmaße, that the last year in 1862 gave the maximum, the penultimate of 1861 the minimum of all seventeen years, and the difference of 1.17 inches seemed so remarkable to me by its size. that I tried to get to the bottom of it. From this the whole following investigation has taken the exit.

First of all, the suspicion arose that the big difference was based on a constant measurement error from opposite direction in both years. Then he could not be expected to find himself again in recruits made and measured elsewhere in Leipzig. So I got the Urlisten of the measurements for the last three years of the entire authority Borna Born, brought them in distribution boards and moved the funds *A*not only for the different vintages, but also for various departments of the Amtshauptmannschaft Borna, and the surprising result was found that without exception the middle dimensions of the years 1860 and 1861 were close in all respects, but the average of 1862 was considerably greater, that is to say in the Throughout those years, a parallel change in the average size of the recruits has taken place throughout the entire head of the central office. This is evidenced by the following table, noting that the term court office generally refers to villages and small patches. From the characters | | and э , which are intended for the comparison of two localities, has not yet been made use of here because it has to be compared several at a time.

I. averages *A* for 20-year-Saxon recruits in different parts of the Amtshauptmannschaft Borna in the years 1860, 1861, 1862nd

(Total *m* = 4736, ***E*** = 1 sax inch = 23.6 mm.)

	A			*m*		
	1860	1861	1862	1860	1861	1862
1) City of Leipzig	69.17	69.06	70.23	616	560	603
2) Gerichtsamt Leipzig I and II	68,85	68.74	69.85	363	326	418
3) City and judicial office Borna	69.39	69.34	70.01	161	169	185
4) Court Office Rötha	69.20	69.12	70.11	79	48	61
5) city and court office Pegau and	69.45	69.10	69.79	157	199	186

Zwenkau						
6) city and court office Taucha and Markranstaedt	68.74	68.93	69.94	109	90	91
7) students	71.47	71.05	71.89	96	111	108
Entire Amtshauptmannschaft	69.26	69.17	70.15	1581	1503	1652

The *A* below of the entire Amtshauptmannschaft are not the means of the *A of* the individual districts, but of the total *m* all in the context, so not singular, but summarily determined (see § 79).

It can be seen from this table that even the movement in the little differentiated years of 1860 and 1861 is parallel in all parts of the territory of the Amtshauptmannschaft Borna, with the exception of No. 6, in that the *A* of 1861 is everywhere smaller than that of 1860; that exception, however, can not alienate at the small *m* of No. 6. Rather, I confess, in which everywhere, do not large *m* to find and small differences in the two years through the present in all other parts of territories parallelism surprised as you to unbalanced him under such conditions. Randomities can neither expect nor find anywhere.

The Leipzigers, among whom, remarkably, the students are not counted, and the students deserve particular attention in the above table in so far as the former originate to a large extent, the latter, of course, from various parts of Saxony. If, therefore, the observed great difference between 1862 and the two previous years could not be sought in a measurement error, then it had to be a more general phenomenon at all.

In order to direct an investigation of this to a part of Saxony that was as different as possible from the one examined so far, I obtained the recruiting measures of the same three years, which were examined earlier, from the Amtshauptmannschaft Annaberg. In fact, the circumstances of the Annaberg Amtshauptmannschaft are very different from those of the Borna people. This is on the northern, those on the southern end of Saxony, this contains flat country with a large city and relatively good food sources, those mountainous terrain merely with small towns and villages and a relatively poor population. The results are included in the following table.

II. Means *A* of measurements in the Amtshauptmannschaft Annaberg in the years 1860, 1861, 1862.

(Total *m* = 3067; ***E*** = 1 inch.)

	A			*m*		
	1860	1861	1862	1860	1861	1862

Cities	68,85	69.04	69.25	369	359	454
Villages	68.99	68.87	69.04	638	565	682
Entire Amtshauptmannschaft. ,	68.94	68.94	69.12	1007	924	1136

If one compares first of all the size movement for the entire A.-H. Annaberg with the for the entire A.-H. Borna, according to the final results of Tables I and II, we find: 1) that for Annaberg 1860 and 1861, or only by taking account of third decimals by an insignificant negative fraction, then by 1861 and 1862 much more substantial, ie by + 0.18 2) that these movements correspond to those of Borna's A.-H. really go parallel; So in both respects a common influence betrays itself. Only the influence for the A.-H. Annaberg much less or more outweighed by influences of opposite kind as for the A.-H. Borna, where the corresponding movements were - 0.09 and + 0.98. However, + 0.18 is still twice as large as the probable difference calculated from the data ± 0.09[3)]. Also between cities and villages of the A.-H. Annaberg finds the parallelism in the years 1861 and 1862 again, and only in the years 1860 and 1861, on which certainly not to be expected, he is missing here.

[3)] The same was found by calculating the probable error in the determination of *A* for both 1861 and 1862 and taking the square root of the sum of their squares.

Insofar as an inferred conclusion can be drawn from previous, still very limited data, it would be true that in the years in question a very general influence of the same direction extended to the size movement over the whole of Saxony, but that this was due to local counter-effects AH. Annaberg only to a greatly reduced extent has been able to come into their own. And that at all in the A.-H. Annaberg other conditions of size development take place than in the A.-H. Borna, it follows directly from the fact that the median measures in that are absolutely smaller than they have found in this.

§ 160. After following the question of parallelism in the preceding merely by consequences of three years each, it undoubtedly had an interest in pursuing it through a long series of years, the claim to prove that parallelism was preferable in the case of the to seek larger movements. In this respect, of Saxon proportions, for comparison, only the Leipzig city measures with the student measurements of 1846-1862, which do not enter into it, have been at my command. and I give in the following table the result of the comparison. After in it for the first year the full value of the A_1 below, the following are merely the movements of each year from the previous one. Keep in mind that the year that accompanies a movement is always the second of the two where the movement takes place. So if z. If, for example, the number - 0.12 stands in the year 1849 - this means that the A_1 of the year 1849 was 0.12 inches smaller than that of the preceding year 1848.

III. Size movements of A_{1} of the Leipzig Stadtmaße and the Studentenmaße from 1846 - 1862 incl.

year	Leipzig	students	
1846	69.19	72.07	
1847	+ 0.10	- 0.37	э
1848	+ 0.28	+ 0.40	||
1849	- 0.12	- 0.79	||
1850	+ 0.37	+ 0.70	||
1851	- 0.18	+ 0.55	э
1852	- 0,11	- 1.02	||
1853	+ 0.52	+ 0.24	||
1854	- 0.04	+ 0.27	э
1855	- 0.28	+ 0.05	э
1856	+ 0.15	- 0.06	э
1857	- 0.28	- 0.41	||
1858	+ 0.44	+ 0.24	||
1859	- 0.89	- 0.96	||
1860	+ 0.04	+ 0.56	||
1861	- 0,11	- 0.42	||
1862	+ 1.17	+ 0.84	||

; ,

It is generally seen first of all that the parallel cases by far outweigh the antiparallel cases; and if the table is rearranged according to the order of magnitude of the measures, the first six motions go without exception according to the measures of Leipzig, according to the students the first ten only with the exception of 1851 parallel to each other, only from there change | and э quite indifferent to what the great relation of *P* to *Q*follows. It is noteworthy that the strongest movement in the students of 1851-52 equals -0.02 only a very insignificant one, albeit equal from the same direction-0.11 in the case of the Leipzigers. Through careful revision I have convinced myself that this is not dependent on a calculation mistake on my part, incidentally, is not to be forgotten that the relatively small *m* each crop year for the students weakens the security of the provision.

Instead of following the movement from one year to the next, as in the previous table, one can follow them from a first to a later one, and derive the results very simply from a table like the previous one, by moving through the years concerned algebraic, ie added with respect to the signs; This is how you get the movements:

year	Leipzig	students
1846-48	+ 0.38	+ 0.03
1848-50	+ 0.25	- 0.09

etc

with six *p,* two *q*. But let us stop at the first, so to speak, elementary table.

This table gives us an opportunity to examine whether and in what circumstances mobility at all is greater on the side of the Leipzig or the students, for which it is only necessary to take the sum of the movements on each side irrespective of omens, which is true for the people of Leipzig , 08, for which students give 7.88; that is, a considerable surplus on the part of the students; which indisputably depends on the fact that the totality of a population from all classes is subject to much more varied, partly destructive influences than the more affluent classes.

On the other hand, if one adds the movements in + and - for each side in particular, one learns how much on each side the variation of the size in + and in - amounted to, which for the Leipzig city measures + 3,07 and - 2 , 01, that is a not insignificant growth in general, whereas the students give +3, 85 and - 4.03, so almost balance between increase and decrease.

It is undeniably expected that in years which give a greater average measure A , even more gigantic results than upper extremes E' occur, generally A and E' are parallel. In addition, this has been confirmed for Leipzigers and students in particular by combining three upper extremes for each year (to better compensate for contingencies); there with 16 movements between 17 years p = 10.5 [4)]; q = 5.5; P = 18.03; Q = 1.23; here at 19 movements between 20 years p = 11; q = 8; P= 21.33; Q = 6.84. Now you should continue to expect that in years with larger A and the lower extreme E , would grow, that grow with increasing average extent even the smallest recruits, and this has, by taking together three minimum dimensions in each year, with students found so: p = 14; q = 5; P = 19.73; Q = 10.99. Very strangely, however, the Leipzigers were just producing the opposite result: p = 4.5; q = 11.5; P = 3.23; Q= 22.62, so that with increasing average measures, the smallest recruits on the whole smaller rather than enlarged. This result, which appears with such great decisiveness, seems strange to me, and at first I can not explain why.

[4)] The 0.5 stems from the fact that there was a motion of zero magnitude between two vintages, where then 0.5 is to be beat both to p and to q .

Further, as above, the mobility of the *A was* compared for Leipzigers and students without regard to the sign of the movements, one can make this comparison also in relation to the extremes. For the sake of comparability with the Leipzigers, I regard the students as above only for the same 17 years 1846-1862, which apply to the Leipzigers, and for the better adjustment of the contingencies, draw not only the movement of extreme extremes, but the means of three each extreme values. This gives the following compilation:

IV. Movement through 17 years .

	For d. Funds from d. totality	For d. Means from 3 minim.	For d. Funds from 3 Maxim.
Leipzig	5.08	27.17	14.67
students	7.88	15.17	16.00

Everywhere, therefore, the arithmetic means A of totality are less mobile than the extremes derived merely as means of three ultimate values, which can not be alienated, and if only the extreme extremes had been taken into consideration, mobility would have been even greater ,

In addition, one can notice again the big difference, between Leipziger and students in the Minimis, while with the Maximis almost agreement between both takes place. For the students, the mobility of the minima is approximately equal to that of the maxima, for the Leipzigers almost twice as large. But all the well is consistent with the earlier [5] together established assumption that the smallest values among the people of Leipzig are abnormal.

[5] [Comp. § 15 and § 128.]

§ 161. Closer to the point, the predominant parallelism that has been pointed out in the foregoing between Leipzig and students can not prove such a difference for different parts of the country, but for a very mixed and for a certain privileged part of the Saxon population, as the people of Leipzig remarked large parts that are actually students from all parts of the country. Insofar as the previously obtained result for different districts of Saxony relates only to very limited space and very limited time, extensive confirmation in both respects had to be desired; what the Belgian measures were doing, for a long time in a consistent manner not only for the whole country, [6] are listed in tabular form. But since vintages with a weak movement of the A or C for a whole country would not lead one to expect any certainty of parallelism for the individual parts of the country, I have made the comparison only for stronger movements, where they can be found for all of Belgium, and for that Movements chosen between following years and epochs:

1) 1852 and 1858;

2) the two five-year epochs 1851-55; 1856-60;

3) two lower epochs of the first of these five-year epochs, ie 1851-53 and 1854-55.

As far as Division 1 is concerned, 1852 and 1858 may be apart, but it does not, remarkably, prevent us from contemplating the movement of size between two distant years; but these vintages are chosen because the first contains the maximum, the last the minimum of *C,* and $g : k$ in a longer succession of vintages, hence the least parallelism of size movement between different parts of the country, if any to be outweighed by unbalanced contingencies and hidden. - The dept. 2), these epochs are distinguished by the fact that the *C* and the $g : k$quite different. - The dept. 3) is a specialization of the first Abtl. from 2).

6) [Exposé de la Situation du Royaume. Bruxelles 1852.]

For 1) only the $g : k$, to 2) the *C* and $g : k$, to 3) the *C* and $g : m$ *are* determined. The determination of these values is given in 2) and 3) summarily for the years in each epoch, after the aggregation of the measures belonging to the same measure intervals, (not singular as means of the determinations of the individual years); the same applies to the final *C of* each epoch, which in the following tables (VI and VII) is in the lowest transverse column (Royaume), with regard to the individual provinces instead of years.

The absolute value of *C* or $g : k$ is given only for the first of the years or periods compared; for the second, the movement again, so that z. Eg in the first of the following tables 1,776 | - 0.182 stands for: 1.776 | 1.594.

Parallelism or anti-parallelism between the various provinces now takes place, according to whether or not the signs of the movements coincide in the same vertical column, which shows that among the 27 movements listed in the following three tables for the nine provinces of Belgium, a single (Liege in the third table) escapes the parallelism (without my being able to find any error in revising the bill), according to which a common influence on the movement throughout Belgium is indubitable.

The magnitude of the parallel movements in the various provinces, however, is very different, and here and there so slight, to make it easy to see that if one wanted to follow the movement between years or epochs, where it is small for the whole of Belgium, then enough antiparallel Of course, even if one had wanted to follow them through all the individual years of each other, as it had happened with regard to the Leipzig and the students, it would always be expected that the parallel cases would be overweight.

In any case, it would not be without interest to make this comparison really so for the provinces of Belgium, where perhaps some characteristic differences might arise for them; and the Documents Statistiques offer sufficient material for that; However, I can not even respond to this, basically very simple, yet far-reaching extension of the investigation.

Incidentally, one can convince oneself from the following tables that the evaluation of the motions according to the $g : k$ or $g : m$ leads to the same results as to the *C* ; Thus, if the above investigation is carried out, the somewhat cumbersome determination of the *C* can be avoided by substitution of previous values.

V. Size movement in the individual provinces of Belgium from 1852 to 1858.

	G : *k*		*m*	
	1852	1858	1852	1858
Anvers	1,776	- 0.182	3249	3796
Brabant	1,832	- 0,558	5490	6208
Flandr. occ. , , ,	1,209	- 0,179	5144	5782
Flandr. or. ...	1,083	- 0.074	6525	7307
Hainaut	1,471	- 0.330	6133	7377
Liège	1,600	- 0,437	3634	4566
Limbourg	2,119	- 0.513	1608	1803
Luxembourg. , ,	2,293	- 0.819	1544	1782
Namur	2,915	- 0.832	2257	2666
Royaume	1.539	- 0.310	35584	41287

VI. Size movement in the individual provinces of Belgium in the following two epochs:

1st epoch: five years, 1851-1833; 2nd epoch: five years, 1856-1860.

	C		*g* : *k*		*m*	
	1.Epoche	2nd epoch	1st epoch	2nd epoch	1.Epoche	2nd epoch
	mm					
Anvers	1645.8	- 3,6	1,584	- 0.097	17368	18382
Brabant	1,650.4	- 9,4	1,767	- .389	29301	30444
Flandr. occ. , ,	1,634.7	- 0.2	1,124	- 0.005	28169	28471
Flandr. or. , , ,	1,633.2	- 1.1	1,075	- 0.027	34648	35483
Hainaut	1,638.1	- 1.8	1,289	- 0,081	33063	36204
Liège	1,647.6	- 6,9	1.602	- 0,259	19842	22206

Limburg. , ,	1,656.7	- 6.3	2,021	- 0.378	8696	8837
Luxembourg. ,	1,658.6	- 9,4	2,167	- 0.460	8279	8823
Namur	1,662.3	- 5.3	2.344	- 0,264	12102	12921
Royaume	1,643.1	- 3.7	1,443	- 0.140	191468	201771

VII. Size movement in the individual provinces of Belgium in the following two epochs:

1st epoch: three years, 1851 - 1853; 2nd epoch: two years, 1854 - 1855.

	C		*g: m*		*m*	
	1851-53	1854-55	1851-53	1854-55	1851-53	1854-55
	mm					
Anvers	1,650.6	- 10,8	0,538	- 0.062	9992	7376
Brabant	1,651.3	- 2.1	0,540	- 0.013	17268	12033
Flandr. occ. , , ,	1,635.8	- 2,9	0,454	- 0.013	16511	11658
Flandr. or. , , ,	1,634.9	- 4.0	0,450	- 0.022	20419	14229
Hainaut	1,639.4	- 3.1	0.472	- 0.020	19088	13975
Liège. , , , ,	1,646.0	+ 3,6	0.513	+ 0.021	11277	8565
Limburg. , ,	1,658.3	- 3,8	0.586	- 0.021	5062	3634
Luxembourg. ,	1,658.9	- 0.7	0.582	- 0.006	4880	3399
Namur	1,664.2	- 4,5	0.608	- 0.012	7117	4988
Royaume	1,644.4	- 3.0	0,505	- 0.017	111611	79857

It would now be desirable to be able to extend the comparison beyond Belgium, for instance to France; However, I do not have sufficient documentation. The Comptes rendus sur le recrutement de l 'armée, however, give annual averages for France for a larger series of years, which are reproduced in a paper by Bischoff, [7] but are subject to the following defects make our purposes completely unusable: in the majority of the series of the vintages the means are so little sharply defined that several times two to four vintages do not differ from each other, and in between individual remedies jump out of line with such values that bill-overs only are too likely.

[7] [On the usability of the results of the recruiting business published in various European countries for the evaluation of the state of development and health of their population Munich 1867 (Verlag der Akademie).]

On the question of a temporal connection between the variations in the size of the recruit.

§ 162. How to understand this question, § 156 is given. First of all, let us examine them in relation to the Saxon dimensions that are available to us, ie the students of Leipzig and the students. The general summation *A of* the first is 69.61, with which the singular coincides. If we now refer to the successive 17 years of 1846 with + or - depending on whether their *A is*above or below this mean, we find the following series of signs:

- - - - + + - + + + + - + - - - +.

At the students the summary *A of* the twenty years is 71,76; with which the singular also agrees. And the sequence of signs hereafter:

+ - + + - + - + + - + + + + - + - - - +.

Now according to the probability calculus of mere coincidence, just as many sign changes would be to be expected as consequences, as one can convince oneself, if one carries out a list of recruiting measures, in which the measures follow at random, and the individual measures also after the series with + or - as long as they are greater or smaller than the A_1 of the list[8]) . In the case of the Leipzig measures, however, the number of strings is 9, that of the change 7, in the case of the students that of the strings 7, that of the change 13. From this there is no temporal connection to conclude, for if such a case exists, then the strings should be decided predominate.

[8] [Strictly speaking, the central value *C should* be subject to the above rule. However, here *A* and *C* do not differ significantly.]

In contrast, the Belgian measures (see Table VIII below) show a very striking correlation. The singular mean *C of* all 33 years from 1843 to 1875 inclusive is 1645.8 mm. In contrast, the entire first 22 years are in minus, the last 11 in plus; and singles out the 33 years in two departments, 16 from 1843 to 1858 incl. *C* = 1641.3 and 17 from 1859 to 1875 with avg. *C* = 1650.0, we obtain the following series of characters in relation to this:

+ + + + - - - - + + + - + - - -;

- - - - - - - - - + + + + + + + +.

Still more, the Belgian measures are not merely a tendency to remain above and below the general average for several years, but also the tendency to rise steadily for a number of years and then to decline again. We find the movements in this respect from 1843 to 1875 following the following signs:

+ + - - - + + + + - - + - - - + - + + + + + - + + - + + - + + +.

The character sequences (sequences of the same signs) are here 17, the symbol change only 14. By mere coincidence, however, twice as many symbol changes would have been expected as consequences. (Thus, as I have convinced myself, if one determines the signs in a corresponding manner on the movements of the randomly successive recruiting measures of the original lists, or in lists of drawn lottery numbers, in which the numbers follow each other by chance, such a determination on the movements of successive numbers.)

In Saxony, the movements of the size of the recruits are traced through 20 vintages, be it A_1, A_2 or C, 5 episodes per 13 bills; so even more change than required, just to apply to accidental.

Since nothing much of a temporal interrelation of variation has been shown in Saxony in the much smaller scale divisions than in the case of all of Belgium, this may prove that this connection is based in the first place on very general causes, which are due to local influences Compensate for larger provincial routes, easy to hide; and it is not only an interesting task to pursue this in other countries as well, but also to investigate with what periodicity of influences the periodicity in human growth is related.

§ 163. I give now the central values *C* for the 33 years 1843-1875, which are derived from me from the original tables; and the associated values $g : k$, where *g is* the number of measures which exceed the interval 1618 to 1643 in magnitude, *k means* the number of those which do not reach it. With these provisions, the total-was *m* all courses 33 years (without inconnue waist) 1304764; the middle *m* thus 39538; the minimum 35584 in 1852; the maximum 41851 in 1860.

VIII. Central values *C* and values *g* : *k* for 19-year-old recruits in Belgium from 1843 to 1875 [9)].

vintage	C	*g: k*	vintage	*C*	*g: k*
	mm			mm	
1843	1,642.1	1,412	1860	1,639.5	1.316
1844	1,642.3	1.414	1861	1,642.0	1,432
1845	1,644.6	1,515	1862	1,642.6	1,474
1846	1,642.3	1,428	1863	1,643.1	1,495
1847	1,640.8	1,357	1864	1,645.1	1,577

1848	1,635.1	1,159	1865	1,647.6	1,694
1849	1,639.6	1,308	1866	1,646.2	1,583
1850	1,641.0	1,340	1867	1,648.7	1,692
1851	1,644.1	1.468	1868	1,653.8	2,022
1852	1644.7	1.539	1869	1,651.27	1,892
1853	1,644.3	1,504	1870	1,651.33	1,876
1854	1,641.2	1,361	1871	1,656.6	1,930
1855	1,641.5	1.370	1872	1,654.2	1,923
1856	1,640.3	1,321	1873	1,659.2	2,233
1857	1,640.2	1.336	1874	1,664.4	2,549
1858	1,637.4	1,229	1875	1,664.5	2,570
1859	1,639.8	1,320			

9) This table differs somewhat from the one I gave in RECLAM's journal in the provisions for the first six years, which resulted from the reduction of 18-year-old recruits to 19-year-olds, because the reduction of *C* in the above table, as well as the *g* **:** *k* is done according to a singular middle-class, whereas in the journal it is done more summarily for the former, only for the latter after a singular middle-education, which makes some entry for comparability. In principle, the former has to be brought forward in our case.

It can be seen that apart from the years 1857 and 1870, the course of the values *g* **:** *k* is parallel with that of the values *C* in the direction of decrease and increase everywhere.

It should be noted that only the values of the years from 1849 onwards are determined by direct measurements of 19-year-old recruits, but the values of the first six years, separated by a dash, but by reduction from measurements of 18-year-old recruits one year previously; so that z. For example, the *C* = 1642.1, which is valid in the table as valid for 19 year old recruits of the year 1843, is derived from a *C* = 1632.5, which was obtained directly from measurements of 18 year old recruits in 1842 [10)]. The following explanation.

10) The values obtained directly for dag *C of* the 18-year-old recruits are in the order: 1632.5; 1632.7; 1635.0; 1632.6; 1631.2; 1625.5.

By the year 1847 inclusive, the recruits were remarkably measured at the age of 18, and were naturally smaller than if they had been measured a year later at 19 years of age. In order to reduce them to this, I have determined the singular mean of the six *C* , as well as *g* : *k of* the age groups of 18-year-old recruits from 1842 to 1847 incl., And the former found 1631.6, the latter 1.033; On the other hand, the corresponding provisions for the 13 years of 19 year old recruits from 1849 to 1861 were searched and found in 1641,2 and 1,373, according to which the *C of* the 18 year old recruits with 1641,2: 1631,6 = 1,0059, the *g* : *k* multiplied by 1.373: 1.033 = 1.329, to be due to the fact that they would have been measured one year later.

The reason I took only 13 years of 19-year-old recruits for comparison with the six-year-olds of 18-year-old recruits to determine the reduction factor while 27 are at their command first was that at the time of making this reduction, I had no more years left; but I stopped here because it would not be appropriate to use too distant vintages for reduction.

If the reduction should occur according to the ratios of the six uppermost *Cs* to the remaining 27 others, it would be undeniably too large a reduction factor 1646.8: 1631.6 = 1.0093, and the general one because of the inclusion of the large values of *C* which are very distant in time singular means of all 33 values of *C* 1646.8 instead of 1645.8.

XXV. Outline and asymmetry of rye

(Secale cereal).

§ 164. With regard to the designations, I note in advance that under panicle I shall understand the fruit-ear, that is, the uppermost part of the stalk, which contains the grains, under the first, second, third, etc., the limbs, or so-called internodes, in order from the top down, under a stalk the whole length: sum of a panicle and joints to a root without them.

In the year 1863, around the 24th of July, from a rye-field to Leutz's care near Leipzig, briefly designated L., a sheaf was harvested from the root to harvest ripe stalks. The majority of them, 217 in number, had 6 members, 138 only 5 members, 10 on the other hand 7 members and 6 of quite stunted appearance only 4 members. The 217 six-membered and 138 five-membered stems of this care, preferably the former, are the subject of the following main study concerning asymmetry relations and asymmetric distribution.

However, it seemed to be of interest whether ears from other locations (around Leipzig) behave in a similar way to those of the Leutzian care in terms of structure, for which a smaller number of stalks had to serve, since otherwise the examination would not have been feasible by me. At the same time smaller bundles of stalks were taken from the following locations around Leipzig with the following content of stalks. At Stünz (St.) July 16: 22 pieces, 20 six-membered, 2 five-membered; on

Täubchenwege (Tbch.) July 20: 24 pieces, 4 six-membered, 20 five-membered; at Schoenefeld (Sch.) 15 July: 22 pieces, 18 six-membered, 4 five-membered. The stalks came from a half-harvested field.

Of all the stalks, the panicle and the individual limbs were especially measured up to the middle of the knot; the total length of the stalk (that is, including the panicle, but without the root) was obtained only by adding the individually measured lengths, since it is practically difficult to carry out the whole To measure stalks in connection, not only because of the often great length of the same, but also because limbs often cling to each other at obtuse angles. According to which the determination of the straw is relatively less accurate than that of its compartments, because the errors of the individual measures partly compensate each other during the addition, but in some cases also add up. Even the lowest limb usually can not be measured accurately, and the determinations in relation to it are of much lower value than for the other limbs because it is usually crippled, so that only above could be measured with the tape measure above; and I would have even left the provisions entirely aside if, on the one hand, a tangible gap had not arisen in the total context of the provisions and, on the other hand, that the provisions above had not, in general, been classified quite well in the context of totality. Sometimes one can be in doubt as to whether the lowest limb is no longer to be expected much more for the root than for the stalk, as occasionally the roots of the lower limb show themselves lowered; but if, from this node, a simple, though stunted, internode runs down to the branched root, it has always been counted as the lowest limb. Even the ripe panicle may be too short due to failure of the lowest grains and the first member next to it is measured too long accordingly; but the length of the panicle was still determined by a small protrusion, better felt by the finger than by the eye, which separates it from the first limb. The awns of the panicle are not measured with.

For measuring a centimeter exactly divided [1)], both measures as uniformly stretched tape measure. Millimeters and sometimes even half a millimeter were appreciated. To give millimeters even on the measuring tape, apart from the fact that the so often repeated sharp eye-sighting would have attacked the eyes too much, brought no substantial advantage, since one can estimate tenths of a centimeter still exactly enough, only that one is faced with the non-uniform estimate of which the dimensions of the recruits and the dimensions of the skull (see Chapter VII) have provided examples. All the sections of the stems, however, were measured once more, after the whole bundle had been measured in groups, not in order to obtain a slight advantage of accuracy both in the middle of both measurements, to detect and improve grosser misunderstandings in the conception and recording by controlling each other independently of each other; Incidentally, which is quite difficult to avoid with so many tedious measures and records as one might think. Of the two measures of the same length then the means could be taken; For the sake of simplicity, however, I preferred to leave the sum of both measures undivided by 2, and all the following details refer to this device, which simply comes to the effect that, as a unit of measure, the half instead of the whole centimeter follows. Of the two measures of the same length then the means could be taken; For the sake of simplicity, however, I

preferred to leave the sum of both measures undivided by 2, and all the following details refer to this device, which simply comes to the effect that, as a unit of measure, the half instead of the whole centimeter follows. Of the two measures of the same length then the means could be taken; For the sake of simplicity, however, I preferred to leave the sum of both measures undivided by 2, and all the following details refer to this device, which simply comes to the effect that, as a unit of measure, the half instead of the whole centimeter follows.

[1] The commercially available tape measures are often inaccurately divided.

§ 165. [In this way the primary tablets for the panicle and the individual limbs of Halime were obtained, from which Table IV in Chap. VII (for the uppermost member of the 217 six-membered stalks) gives an example. From these, the following tables were derived.]

Since the unit of measure ***E*** for the rye is ½ cm everywhere, I omit the following a special mention of the same.

I. Value of A_1 for panicle and limbs, depending on different number of limbs and different location, the total length of the culm is set equal to 100.

	7 members	6 members			5 members	
	L. (10)	L. (217)	St. (20)	Sch. (18)	L. (138)	TBCH. (20)
Panicle	5.8	5.9	7.1	5.7	6.5	5.0
1st member	27.5	31.4	31.6	33.7	35.4	34.6
2nd member	23.6	26.1	25.3	28.7	28.5	28.8
3rd member	15.6	16.3	15.7	15.6	16.0	16.9
4th member	12.3	11.8	12.0	10.0	10.2	10.5
5th member	9.3	6.7	6.8	5.1	3.4	4.2
6th member	5.2	1.8	1.5	1.2	-	-
7th member	0.7	-	-	-	-	-
Absolute values of A_1 for the	318.9	275.2	344.7	286.9	261.1	222.1

whole Straw						

II. Values of η : A_1.

	7 members	6 members			5 members	
	L. (10)	L. (217)	St. (20)	Sch. (18)	L. (138)	TBCH. (20)
Panicle	0.285	0.212	0.234	0.183	0.217	0.184
1st member	0,119	0.115	0.116	0.105	0.108	0,101
2nd member	0.106	0,117	0.114	0.106	0.126	0,101
3rd member	0,111	0,119	0.168 [2)]	0,099	0,128	0.144
4th member	0,128	0.141	0.094	0.135	0.201	0.177
5th member	0,157	0.253	0,179	0.312	0,407	0,490
6th member	0.164	0.487	0.542	0.576	-	-
7th member	0,241	-	-	-	-	-
Whole stem. ,	0.083	0,099	0,076	0.093	0.104	0,089

[2)] 0.168, although proved correct by revision, is to be regarded as abnormal, since otherwise everywhere the η : *A of* the third term is smaller than that of the fourth.

III. Elements of the 217 six-limbed stalks of Leutzian care after primary panel.

	panicle	1. Eq.	2. Eq.	3. Eq.	4. Eq.	5. Eq.	6. Eq.	stalk

A_1	16.2	86.5	71.8	44.9	32.5	18.4	4.9	275.2
G_1	15.8	85.5	71.0	44.2	31.9	17.4	4.0	272.8
E ,	7.5	42.9	38.9	19.1	15.0	6.0	0.6	147.9
e'	27.9	112.2	99.8	61.9	48.0	34.0	19.0	352.6
U	- 5	+ 25	+ 10	+10	- 3	- 15	- 33	+ 13
$U' - U$,	+ 3.0	- 17.9	- 4.9	- 8,8	- 2.0	+ 3,2	+ 9.8	- 49,9

IV. Elements of the 138 five-limbed stalks of Leutzian care according to the primary panel.

	panicle	1. Eq.	2. Eq.	3. G1.	4. Eq.	5.Gl.	stalk
A_1	16.9	92.4	74.4	41.8	26.7	8.9	261.1
G_1	16.3	91.5	73.4	41.2	25.8	7.6	258.8
E ,	7.0	53.5	34.1	19.5	6.3	1.6	158.7
e'	33.4	119.4	96.4	62.4	41.8	22.0	330.9
u	- 2	+ 14	+ 8	+ 8	+ 4	- 14	+ 10
$U' - U$,	+ 6,6	- 11.9	- 18.3	- 1.7	- 5.3	+ 5.8	- 32.6

§ 166. The results of the most general interest, which can be drawn from the above tables, seem to me to be the following two.

(1) The fact that there are certain legal relationships in the rye of the species which may be regarded as characteristic of rye and which may well give rise not only to examining the different cereals and gramineae thereafter in the interest of their comparative character, but also the influence of external circumstances, such as the soil condition and annual weather to study.

2) That this results in decisive evidence of the existence of a substantial asymmetry and a basis for examining its laws.

Let us first investigate the former interest of the investigation.

It may be questioned whether the variations which the individual rye-stalks show in regard to their length and their relations of arrangement depend rather on a chance difference of the seeds or the nature of the soil from which each one is rearranged, probably of both causes. without being able to decide empirically so far. In any case, the following collective relationships take place.

1) Despite the fact that the average length A_1 of the whole stalks fluctuates between 344.7 and 222.1, depending on the location, as indicated by the information given in

Table I, the ratios of the links (according to their arithmetic means) to the total length are independent of which, and only with the number of terms, can be regarded as variable; in short, they can be considered as constant and thus characteristic for the rye given the number of terms. Table I contains the supporting documents, provided that all limbs and the panicle are reduced by the ratio of the straw (equal to 100). Since apart from Leutzsch with m = 217 and 138 the other locations only one m = 10; 18 and 20, I would not have thought that in the small $m$conditional uncertainty, the correspondence of the relative member lengths for a given number of members could have gone as far as it is the case. Only at Schönefeld (with m = 18) are there some larger differences from the other sites for the six-limbed stalks; but compare for the six limbs. Halms the surprising attunement of the relationships between L. (217) and St. (20) in the very different total lengths 275.2 and 344.7; as well as the not less remarkable for the five-membered. Stems between L. (138) and Tbch. (20) at the different total length 261.1 and 222.1. Yes, even Sch. fünfgliedr. with m = 4 is odd with it, and only Tbch. sechsgliedr. with m = 4 and L. viergliedr. with m= 6 show not insignificant deviations; but comparisons with such small m can not be decisive at all and have therefore been omitted in the previous table. Incidentally, it would have been more appropriate at all to take the individual limbs into consideration in proportion to the sum of the limbs, without panicles, as panicles, as has happened here.

2) If you compare the columns for the seven, six and five members. Halms of Tab. I, it is generally found that with descending in this number of limbs, the three first limbs increase in proportion, but the last decrease. In short, if the number of links decreases, the upper links extend and the lower links shorten in proportion to the total length. For the panicle, no particular rule is visible in this regard.

3) If one raises the question as to whether the assertion made by ZEISING and repeatedly accepted confirms that in nature the irrational ratio of the golden ratio, ie, exactly 100 **:** 162, plays an excellent role in nature If this is not the case according to Table I, then the relation of the successive members to each other will be quite variable. Just as little does a tendency to simple rational relations seem to exist.

4) The simple mean ερρορ or the simple mean fluctuation $\eta = \sum\Delta : m \,.\, A$ decreases in absolute value from the top to the lowest limb, for which I have not attached a table. But as the value of A decreases in this direction, the question αρισεσ as to what happens to the relative value $\eta : A = \sum\Delta : m\,A$, or the relative fluctuation in this respect, which is to be judged according to Table II. Here is the remarkable thing that $\eta : A$of the two to three supreme limbs, neither according to the ordinal number of these limbs (whether first, second limb, etc.), nor according to the nature of the stalks (whether seven, six, or five limbs), nor, at last, varies considerably according to location, only that in the seven- and six-membered stalks the noticeable constancy extends to the three [3)] , in the case of the five-membered stems only to the two uppermost members. But, as we descend to deeper terms, not only does $\eta : A$generally grow with the depth of the members at equality of position and number of members, but also changes in equality of atomic number after these two

moments. The η : *A*the panicle is considerably larger everywhere, on average about twice as large as that of the first member, whereas the η : *A of* the whole branch is smaller than that of any department; which is easy to understand.

[3] The value 0.168 for the third term Stünz is recognizably abnormal, without being due to accounting errors, since it is followed by the smaller value 0.094 in the fourth term.

Since in the values of η : *A* of Table II the η ισ uncorrected, by applying the correction (see § 44), the stated values would actually have to be increased for the following values *m* in the following proportions *v* :

m	10; 20; 138; 217
v	1.054; 1.026; 1,004; 1,002.

But it is easy to see that this would not change anything in the conclusions drawn.

§ 167. After this I come to the part of the investigation which relates to the relations of asymmetry; to which only the data obtained from the Leutzsch site with 217 six-membered. and 138 five-membered Halmen grant a sufficient *m* . Even a $m = 217$ is certainly not big enough for the influence. unbalanced coincidences to a desired degree depress [4] , but it will be seen that when the required reduction and sharp treatment, the calculation results are in excellent agreement with the sets of collective asymmetry; but without any reduction already give the values of $u = \mu' - \mu,$ and $U' - U,$ (of which $U' = E' - A$; $U, = A - E,$) in Tables III and IV the proof that essential asymmetry exists here.

[4] [In fact, the probable value *V of* the difference $u = \mu' - \mu,$ rel. A_1 assuming essential symmetry according to § 98 on the basis of the formula $V = \pm 0.6745$ equal to ± 10.]

Should namely essential symmetry of deviations bez. *A* , the difference *u* between the two deviation numbers μ', $\mu,$ and the difference $U' - U,$ between the two extreme deviations, which must be shown in Tab. III u. IV not given, but when $U' = E' - A$ and $U, = A - E,$ are easy to find out of it, depend only on unbalanced contingencies and change randomly between limbs of the blades according to size and sign. But let's follow the difference *u*downwards through the series of limbs, we see the positive value of the first limb continually decreasing in size, and of a certain limb (for the six-limbed stalks from the fourth - for the five limbs, only at the fifth limb itself) negative folding. We do just as with the differences $U' - U,$, we find the Corresponding with the opposite sign, except that here also with the sechsgliedr. Halmen the envelope begins only at the fifth link. At the same time, these tables give the opportunity to prove the general proposition (§ 33; 142) that $U' - U,$ which has opposite sign of $\mu' - \mu,$ has only a very small one*u* and U' -

U, can suffer an apparent exception through unbalanced contingencies, of which the example of the fourth member of the six-limbed is also to be found here. Finds straws. For the panicle is in the six- as fünfgliedr. Straws *and* negative, $U' - U$, positive; for the whole straw the first value is positive, the latter negative.

It would now be very interesting to investigate whether the determined pronounced legal course of u and $U' - U$, which here just for a single location (Leutzsch) and the weather in a particular year (1863) for sufficiently large m located It is also found in other locations and other annual weather conditions, since it is very possible that other locations and weather conditions during the growth of the stalks have different circumstances in this regard. Now I have the data for other locations (St., Tbch., Sch.) Before, but only with a $m$from 18 to 20, which is much too little to expect certain results, but in order to justify one guess at least, St. and Tch., both with $m = 20$, have examined the course of their u, taking into account the following table received results.

V. A_1 and u for the locations Tbch. and St., both with $m = 20$.

	A_1		u	
	TBCH. 5 gl.	St. 6 gl.	TBCH.	St.
Panicle. , ,	11.2	24.5	- 6	- 2
1. Eq. , ,	76.8	108.9	- 2	± 0
2. Eq. , ,	63.9	87.2	± 0	+ 2
3. Eq. , ,	37.6	54.1	- 2	- 2
4. Eq. , ,	23.3	41.4	- 6	+ 2
5. Eq. ,	9.3	23.4	- 2	± 0
6. Eq. , ,	-	5.2	-	- 4
Halm. , ,	222.1	344.7	- 6	+2

According to this, however, one may presume with reasonable certainty that the location is of considerable influence on the course of the u, and hence the asymmetry of the rye, since for Tbch. all u *are* negative or zero, for St. indefinitely change in size and sign [5].

[5] [It should be noted, however, that here the probable value of u is given the assumption of essential symmetry. A_1 from the formula $V = \pm 0.67$ (see § 98)

equals ± 3, according to which only three of the above thirteen values exceed the probable value *V.* It is thus indeed possible to assume an overgrowth of purely accidental asymmetry, which by no means precludes that for Tbch. and St. with larger $m$similar laws can occur as those observed for L.]

§ 168. For the results hitherto, only the primary plates were omitted, which, however, do not permit a sufficient determination of the densest value, calculation of the distribution dependent thereon, and in general investigation of the relations relating to *D.* We now proceed to reduced tables, which will henceforth be limited to the Leutz material, namely the six-membered one with $m = 217$.

[But also of this material should be considered only the five upper limbs. For they suffice for the proof of the asymmetrical laws of distribution, and permit a sufficient, corrective control of the course of asymmetry that emerges in Plate III. It is moreover advisable to refrain from the panicle and the lowest limb, for the results given above (§ 164) would have a doubtful value. Accordingly, let us give the *z* values of the first five terms for a reduced $i = 4$ ***E*** in the arbitrarily chosen reduction position, and directly add to the observed values the calculated values as given by the two-sided GG. Immediately following this are the elements used in the calculation:

VI. Reduced table of the 217 six-membered stalks (L.).

$i = 4$ ***E*** ; $m = 217$.

1st limb 2nd limb 3rd limb 4th limb 5th limb

	z			*z*			*z*			*z*			*z*	
a	Obs.	calc.	*a*	Obs.	calc.	a	Obs.	calc.	a	Obs.	calc.	a	Obs.	about
44	1	1	38	1	1	18	1	0	15	3	1.5	3	0	2
48	1	1	42	1	1	22	1	0.5	19	5	6	7	11.5	10
52	1	1	46	1.5	3	26	2.5	2	23	12.5	17	11	29	28
56	2	2	50	6.5	5	30	4.5	6	27	38	36	15	48	50
60	4	3	54	6.5	8.5	34	16.5	15	31	55.5	53.5	19	**63.5**	56
64	6	6	58	15.5	13	38	20.5	29	35	**57.5**	54	23	38	41
68	8th	9	62	17.5	18.5	42	43.5	42.5	39	31.5	34	27	15.5	21
72	9	13	66	25.5	24	46	**58.5**	49	43	11	12	31	8th	7
76	21.5	17	70	29.5	29	50	39	41	47	3	3	35	3.5	2
80	15.5	22	74	30.5	32	54	19	22						
84	24	25	78	**32**	32	58	7	8th						

88	**33.5**	28	82	25.5	25	62	4	2						
92	27.5	28	86	16	15									
96	23.5	24	90	6.5	7									
100	18.5	18	94	0.5	2									
104	13.5	11	98	1.5	1									
108	4	6												
112	3.5	3												

VII. Elements of the 217 six-membered stalks (L. after reduced slab.

	1st member	2nd link	3rd member	4th member	5th member
A_2	86.52	71.69	44.83	32.39	18.38
C_2	87.85	72.52	45,30	32,60	18.26
D_p	90.58	76.73	46.23	33.46	17.96
D_i	88.45	76.75	45.74	33.29	18.51
u	- 45	- 65	- 27	- 24	+10
e ,	11.82	10.98	6.28	5.33	4.60
e'	7.76	5.94	4.88	4.26	5.02
p	0.67	0.84	0.66	0.80	0.71

The comparison between theory and experience shows a sufficient agreement, which is all the more satisfactory since the underlying m = 217 is relatively small. In particular, it may be noted that the second term corresponds well to the requirements of the theory, in which, of course, no distinguishing feature is to be found with respect to the other terms, but only a contingent coincidence with the reduction stage and the reduction position just selected. Thus, the two-sided GG on the rye stalks proves itself.]

[At the same time, the existence of essential asymmetry is questioned. However, in order to check the inferences regarding the decrease and reversal of the asymmetry for descending limbs suggested by the regular course of the u - values in Tables III and IV, it is indicated with the *u of* the table referring to A_1 III the corresponding bez. D_p applicable *u*above table. This comparison teaches that here the second member has the maximum value in place of the first, and the reversal of the asymmetry occurs only at the fifth member instead of at the fourth, and that in general the fluctuations between the successive members are differently distributed and stronger than there. If one now asks which values are to be regarded as authoritative, then one must take into account that although always a u value bez. A a, with the ratios (D - C) **:** (C - A) growing, relatively large u value bez. $D$of opposite sign, but that the choice of reduction and reduction position influences the position of the values D , C and A , namely that of D , more strongly than those of C and A , as from the comparison tables of the elements for different reduction stages and Reduction situations in chapter VIII. This explains the sharper fluctuations of the ***u*** in comparison to the quieter course of the u. Nevertheless, a final judgment on the asymmetries is rather on the ***u*** than on the u to found. For the latter give only a clue to determine whether and in how far those with essential symmetry bez. A expected u values are exceeded by the observed ones; on the other hand, assuming essential asymmetry, D_p is the most probable value and, accordingly, the probabilities p and $q = 1 - p$ for upper and lower deviation in the ratio of the observed mean deviations e 'and e , must be assumed, while a corresponding assumption for the deviations $A$is not allowed. They are therefore in agreement with the details of the addition to Chap. XIV (§ 101) equals the probable bounds of u :

and to set on the basis of the proportion $p : q = e' : e$, to calculate, according to which in the present case, the value ± 10 rounded for each of the five members as top and bottom likely limit, from those indicated in the table likely ***u*** - Calculated values, results. From this, however, it follows not only that each limb, considered individually, has essential asymmetry, but also that the fluctuations between the successive limbs, with the exception of those between the third and fourth limbs, are to be regarded as essential. In this case, however , the uncertainty in the determination of the uncertainty due to the smallness of m and the choice of the reduction position is$D_{p\ is}$ not considered, it is advisable not to put too great weight on absolute values of the observed ***u*** and only in general the tendency to decrease of asymmetry at descent in a number of links and to reverse asymmetry at the lower links stress.]

§ 169. [Finally, the question arises as to whether the relations of the rye members are subject to collective treatment. This interest is served by the following two tables which, for the ratios of the first and second terms and of the second and third terms, are reduced tables for comparison between observation and calculation, as well as each time the values of the elements on the basis of the logarithmic law of distribution. The three successive smallest and largest values of the ratios of the first and second terms are 0.64, 0.98 and 1.00, respectively; 1.50, 1.97 and 2.11 on the other hand. The corresponding values for the ratios of the second and third members are 1.12, 1.15 and 1.16 on the one hand; 2.22, 2.42 and 2.63 on the other. The with aIn the first case, logarithms which are to be designated thus hold between the limits-0.19 and + 0.32; in the latter case, between the limits of 0.05 and 0.42. At a reduced $i = 0.02$ this leads to the following values:

VIII. Relations of the three uppermost limbs of the 217 six-limbed stalks (L.) and their elements.

$i = 0.02$; $m = 217$.

1st member **:** 2nd member

a	Z Obs.	Z calc.	
- 0.19	1	0	***G*** = 0.080
- 0.03	0	1	***C*** = 0.079
- 0,01	1.5	3	$\boldsymbol{D}_p$ = 0.076
+0.01	11.5	9	$\boldsymbol{D}_i$ = 0.080
+0.03	15	21	***u*** = + 13
+0.05	35	**34**	*e* **,** = 0.030
+0.07	**47**	**43**	*e* '= 0.034
+0.09	**47**	41	*p* = 0.75
+0.11	30	31	G = 1.202
+0.13	16	19	*C* = 1,199
+0.15	7	10	$\boldsymbol{T}_p$ = 1.191
+0.17	4	4	$\boldsymbol{T}_i$ = 1.202
+0.19	0	1	
+0.29	1	0	
+0.33	1	0	

2nd member **:** 3rd member

a	z	

	Obs.	calc.	
0.05	1	1	***G*** = 0.206
0.07	5	2	***C*** = 0.206
0.09	3	5	
0.11	8th	8th	D_p = 0.206
0.13	14	13	D_i = 0.210
0.15	17.5	19	
0.17	23.5	24	***u*** = 0
0.19	26	28	
0.21	**37**	29	*e* , = 0.048
0.23	26	36	*e* '= 0.048
0.25	17	22	*p* = 0: 0
0.27	14	16	
0.29	9	11	*G* = 1.607
0.31	9	7	*C* = 1.607
0.33	2	3	T_p = 1.607
0.35	3	2	
0.37	0	1	T_i = 1.622
0.39	1	0	
0.41	1	0	

Noteworthy is the low degree of asymmetry, which is completely absent for the ratio of the second and third term, and would not occur until the fourth decimal of the main values ***G*** ,***C*** , and D_p . However, taking into account the fourth decimal place would not change the theoretical distribution of the *z* over the individual intervals, since it would only influence the fractions of the *z* . The values ***G*** are equal to 0.081 as determined from the primary tables for the ratio of the first and second terms, and equal to 0.205 for the ratio of the second and third terms. The extreme ***a*** for the first and second terms, the distributional calculation turns out to be decidedly abnormal.]

XXVI. The dimensions of the gallery paintings.

§ 170. [Im XXI. Chapter has already been a K.-G. taken from the dimensions of the gallery paintings and presented as an example in the interest of comparison between the arithmetic and logarithmic treatment. Here, the measures of the original lists, as given by the catalogs cited there, served as a direct basis for the preparation of the reduced tables of distribution, as well as for logarithmic and arithmetical reduction. Here are the results of the detailed investigation, which regarding the dimensions of the various gallery paintings from the point of view of collective asymmetry in the appendix sections to the "preschool of aesthetics"; has been conducted and the arithmetically reduced distribution tables listed there are partly based on a logarithmic treatment. At the same time, the latter can serve as a proof that the arithmetically reduced tables can still provide a sufficient documentary support for logarithmic treatment, even if, as in the present case, the end department of the larger measures deviates from a limit is summarized as a remainder and its extent can only be determined from the specified extreme values.]

[I now take from the designated source [1)] first the information about the state of the investigation (§ 171) and further (§ 172 and 173) the distribution tables and the tables of the elements together with the discussions to be made, then (§ 174) Success of logarithmic treatment to show four examples. Finally, I share (§ 175) again from the preschool of the aesthetics information about the relationship of height and width and about the surface area of the gallery paintings with]

[1)] [preschool of aesthetics; 1876. Part Two, p. 275 flgd.]

§171. As a class of pictures religious, mythological, genre, landscape and still life pictures are distinguished:

a) Religious pictures, ie pictures with Old Testament and Christian religious content. For this purpose, not only compositions with several figures were counted, but also individual heads and figures, such as Christ's heads, images of saints, representations of martyrs, even landscapes with sacred staffage, so that this class is actually a poorly defined hodgepodge; Therefore, a very irregular distribution of measure and number took place in it.

b) Mythological, ie pictures with a content from the Greek and Roman world of gods and heroes, correspondingly broad, therefore also poorly distributed.

c) genre pictures, in the usual sense, without war and hunting scenes.

d) Landscapes, including marines, but without port and city views.

e) Still lifes, images of dead objects (apart from the architecture excluded), such as compilations of food, utensils, and flower and fruit pieces, with the exception of those which include human figures, but including those in which animals incidentally occur.

Worldly historical pictures, architectural pictures, portraits, and pictures that are not understood in previous classes are not examined. Everywhere are excluded fresco and wallpaper pictures, diptychs and triptychs and such panels, in which various representations were contained in delimited departments.

Of course, several doubts could arise as to whether a picture should be included as genre picture under c) or left aside as a secular historical picture, whether a picture should be taken as a landscape under d) or left aside as a mere cattle piece, etc .; and certainly others could have classified the dubious cases a little differently. However, this does not matter much, because the uncertainty always concerns relatively few pictures, so that the conditions can not be significantly involved. A very sharp separation principle can not be set up at all; I went to the apercu of the predominant impression of the picture designation in the catalogs.

In many cases, two or even a series of related images of the same format are listed one after the other in the catalogs. So in the third part of the Louvre catalog: École francaise p. 342 ff. 525 to 547 under the common title: "Les principaux traits de la vie de St. Bruno", 22 pictures of LE SUEUR in front, which, with the exception of no. 533, all the same dimensions h= 193; b = 130 cm.

The question arose whether in such cases all specimens as a single one should be included in the distribution table and charged only once or as often as they occurred.

If this were to be the point, but which would have little interest in determining the actual mean values of the images of a given kind contained in given galleries and the factual distributive relations, then of course only the latter method could be observed; but since it was not to be expected that in other galleries the same dimensions would return on average in the same ratio, in this way one would receive an inappropriate contribution to the general determination of the mean, and thus find the general distributional conditions considerably altered. Thus, the following numbers of religious images were found in the following size intervals of height:

intervals	z
cm cm	
179.5 - 189.5	91
189.5 - 199.5	89
199.5 -	93

which numbers closely match, as expected at adjacent intervals. But here all 22 SUEUR pictures of 193 cm height are only counted twice, if you had to calculate them 22 times, you would have 91 instead of the consecutive numbers; 89; 93 obtained: 91; 109; 93; which would have made the distribution very irregular. Correspondingly in other cases. Since, however, a large number of related images of the same dimensions presupposes a certain strong preference for these dimensions, and thus takes on an increased weight, I have decided, briefly and approximately, to find all cases where two or more related images of the same dimensions were present to be counted twice, but not more than twice, in the distribution board.

Therefore, if the following is the total number of images examined at 10558, this figure is not strict in so far as only a mere two are taken into account from a larger number of related images of equal dimensions, but on the other hand landscape images, in which religious and my-thological staffage occurs, both in the landscape paintings as religious or mythological images, so are recorded twice. However, since the influence of both circumstances is not at all considerable and, moreover, from the opposite direction, the above figure remains sufficiently close enough.

There are only a gallery images, namely twenty-two public galleries [2)] measured or rather those given in the catalogs gallery dimensions, walking on the image size in the lights of the frame, and uses the comparability, all have been reduced to metric measure.

[2)] Used catalogs.

Amsterdam. Beschriving the Schilderijen ops Rijks Museum te Amsterdam 1858.

Antwerp. Catalog of the Musée d'Anvers, without year.

Berlin. a) List of the Royal Collection of Paintings Museums to Berlin 1834.

b) List of the collection of paintings of Consul Wagener 1861.

Brunswick . PAPE, d. Gemäldesamml. d. Heart. Museum of Brunswick 1849.

Brussels. FÉTiS, Catalog descript. et histor. you mus. roy. de Belgique 1804.

Darmstadt . MÜLLER, description d. Gemäldesamml. in d. Big heart. Mus. to Darmstadt.

Dijon. Notice of the objets d'art exposés au Mus. de Dijon 1860.

Dresden. HÜBNER, Verz. The Königl. Picture gallery to Dresden 1856.

Florence. CHIAVACCHI, Guida della R. Gall. del Palazzo Pitti 1864.

Frankfurt. PASSAVANT, d. D. public. -equipped kitchenette. Works of art. d. Städel Art Institute 1844.

Leipzig , a) d. D. Artworks d. Urban Mus. to Leipzig 1862.

b) d. d. Löhr's collection of paintings at Leipzig 1859.

London. The National Gallery, its pictures etc. Without year.

Madrid. PEDRO DA MADRAZO, Catalogo de los quadros del real Mus. de Pintura y Escultura 1843.

Milan. Guida per la regia Pinacotheca di Brera.

Munich. a) d. d. Gem. In d. Royal. Pinakothek to Munich 1860.

b) d. d. Gem. D. new royal Pinakothek in Munich 1861.

Paris . VILLOT, Notice of the tabl. exp. dans les gal. du mus. imp. you Louvre 1859.

Petersburg . BALANCES, The painting. in the Kaiserl. Hermitage to St. Petersburg 1864.

Venice. Catalogo degli oggetti d'arte esposti al Publico nella L. Roy. Accad. di belli arti in V. 1864.

Vienna. v. MECHEL, d. Gem. Of the KK picture collection 1781.

As a unit of measure therefore follows the following without exception, the centimeter.

§ 172. The investigation has extended to the classes indicated above; yet, for reasons given, the religious and mythological ones have been brought to too few regulations. In each class, however, two divisions are distinguished; namely, images in which the height *h is* greater than the width *b* , and those in which the reverse applies; the former with $h > b$, *the* latter with $b > h$. Between these two sections, the very rare square pictures are distributed equally, as they were presented, [3]), However, provisions are also drawn from the aggregation of the two departments, which apply mutatis mutandis to the *h* and *b of the* same.

3) In any case, this is more correct than completely attributing it to one department or the other, because in the pictures listed as square, one, and soon the other dimension will soon be slightly larger than the other, except that the measurement is very small Differences not taken into account.

Hereafter z. B. *h* ; $h > b$ Height of images whose height is greater than the width, and *b* ; $h> b$ Width dimensions of images whose height is greater than the width, etc., finally *h* ; comb. or *b* ; comb. Height or width dimensions of images of the combined divisions $h > b$ and $b > h$.

The primary distribution boards of the classes and departments taken up in studies, of which $i = 1$ cm, naturally have a large extent and are subject to strong irregularities. The following sample must suffice to give an idea of their appearance:

I. Sample from the primary distribution panels.

(Genre: *h* ; $h> b$).

a	z	*a*	Z
29	13	41	17
30	15	42	14
31	13	43	14
32	20	44	12
33	21	45	15
34	9	46	10
35	17	47	17
36	13	48	10
37	22	49	12
38	26	50	4
39	8th	51	12
40	9	etc	

In order to limit both the expansion and the irregularities, it is necessary to go to reduced slabs and use the same as i = 10 cm.

Here are the reduced panels for both genre and landscape departments and for $h > b$ still life. The total number *m of* the copies of each class and department is given below. Many numbers in the table show a decimal 0.5 attached. This is due to the fact that numbers falling on the limit of an interval itself have been assigned by the method of the divided z , half the one, half the other of the intervals thus diverted, which carries half an unit for odd numbers. If one wants the measures of h or b for the combined $h > b$ and $b> h$you just need to add up the measures of both departments.

II. Arithmetically reduced distribution panel for genre, landscape and still life.

i = 10; ***E*** = 1 cm.

A	genre				landscape				still life	
	h> b		h		h> b		h		*h> b*	
	H	*b*	*H*	*b*	*H*	*B*	*H*	*b*	*H*	*b*
5	-	5	-	-	-	-	6.5	1.5	-	-
15	30.5	88	23	6	2	8.5	66	18	-	4

25	133	**190.5**	90.5	38.5	17.5	23	200.5	90	10.5	16.5
35	**161**	167.5	109	78.5	26.5	**53.5**	**278.5**	166	24.5	44
45	127.5	100.5	**114.5**	80.5	32.5	40	257.5	189	**50.5**	45
55	75.5	62.5	79.5	75.5	22	33	219	168	27	**51**
65	70	58.5	65.5	**86**	**41.5**	21	165	**202**	31.5	45
75	47	31.5	40.5	34.5	25	13.5	139	135.5	29	32
85	39.5	18	28	63.5	8.5	20	79	139.5	38	22
95	20.5	21	33	36.5	20.5	14	93	125.5	23.5	17.5
105	12.5	8th	17	26.5	13.5	8.5	69	78	17.5	12
115	11.5	10	25.5	29	10	9	45	63	14.5	2.5
125	12.5	2.5	24	24	6.5	5	36.5	58.5	16	6.5
135	12.5	1.5	11	12	7.5	2	28.5	71.5	5.5	3
145	7.5	5	15	19	7.5	10	19.5	39	2	1
155	11	2.5	6	9.5	5	9.5	29	33.5	1	3
rest	3	2.5	20	82.5	36	11.5	62.5	215.5	17	3
m =	775	775	702	702	282	282	1794	1794	308	308

It can be seen that the distribution essentially follows the same course everywhere. Everywhere there is a principal interval, in which the measure is a maximum, from whence the measures decrease rapidly on both sides, and the principal interval lies much closer to the upper end of the tablet, which begins with the smallest measures, than to the lower which would be even more conspicuous, if not the numbers for all measures over 160 cm in bulk and bow (as rest) would be summarized. Herewith the board offers a particularly interesting example of a K.-G. It shows that the progression of the values from the main interval down to both sides has come very close to a regular one. Here and there, of course, especially with genre *b* ;*h* landscape *h* ; *h*> *b* and *b* ; *There* are still strong irregularities and there is no lack of small numbers in the lowest part of the table; but it can be assumed that these would disappear altogether or diminish greatly if a much larger number of the specimens had been available, and the more so as they balance each other out, the greater the intervals at which the measures are summarized.

The genre, landscape, and still-life pictures show a similar course to the religious and mythological ones, except that in these classes, undoubtedly because of the unfavorable combination of the pictures calculated below, some very large irregularities remain in progress, barely magnified *m* expected to offset, so these classes are not suitable for testing the distribution laws and have not been so far worked out of me than the others. Even for still lifes *b*> *h* relatively stronger irregularities have remained, than that a complete study would have been worthwhile.

§ 173. However, a more detailed insight into the proportions and asymmetry of the gallery paintings can be obtained only from the following information about their elements, for the calculation of the original distribution boards were based.

III. Elements for genre, landscape, still life, religious and mythological after primary board.

$\boldsymbol{E}$ = 1 cm.

			m	A_1	G_1	C_1	H	$\eta:A_1$	u
genre	h> b	H	775	54.4	46.7	44.6	24.4	0.45	- 197
		b	775	43.6	37.4	35.8	19.6	0.45	- 191
	$b > h$	H	702	63.8	53.8	51.4	30.3	0.47	- 182
		b	702	86.8	72.0	67.8	42.7	0.49	- 196
	comb.	H	1477	58.9	50.0	47.8	27.4	0.47	- 379
		b	1477	64.0	51.0	49.4	34.7	0.54	- 437
landscape	$h > b$	H	282	88.1	73.3	70.1	44.1	0.50	- 60
		b	282	69.1	58.7	54.6	25.3	0.37	- 75
	$b > h$	H	1794	64.7	54.5	53.3	30.3	0.47	- 426
		b	1794	90.3	75.2	74.4	43.6	0.48	- 436
	comb.	H	2076	67.9	56.7	55.7	27.4	0.40	- 520
		b	2076	87.4	72.8	71.2	34.7	0.40	- 522
Still life	$h > b$	H	308	80.6	72.6	73.0	29.0	0.36	- 42
		b	308	62.2	57.7	58.9	21.9	0.35	- 34
	h	H	204	71.0	60.1	55.7	-	-	- 54
		b	204	95.2	83.5	76.6	-	-	- 60
	comb.	H	512	76.8	67.3	67.3	-	-	-
		b	512	76.4	66.8	65.0	-	-	-
Religious	$h > b$	H	3730	135.4	-	109.5	75.5	0.56	- 804
		b	3730	107.0	-	76.0	44.5	0.42	-1274
	h	H	1804	111.6	-	96.1	56.6	0.51	- 316
		b	1804	156.1	-	131.5	80.6	0.52	- 388
mythological	$h > b$	H	350	141.7	-	133.3	66.1	0.47	- 30
		b	350	103.8	-	95.0	55.8	0.54	- 42
	h	H	609	116.9	-	104.9	60.0	0.51	- 89
		b	609	158.0	-	146.1	74.2	0.47	- 57

First of all, determinations about the relative frequency of the occurrence of images of a given class and department in galleries can be deduced from the values *m* in the previous table, although it must be remembered that the ratios of this frequency differ greatly according to the individual galleries; the special statistics in this regard would only cost too much space in proportion to their interest. If we keep to the overall result of the twenty-two galleries, then (without distinguishing the sections *h*> *b* and *b*> *h*) according to the combined values, the five classes studied in relation to the frequency of the images: Religious, Landscapes, Genre, Mythological, Still life. The relationship of landscapes to genre in particular (2076**:** 1477) something exceeds the ratio 4 **:** 3.

Of genre pictures, those whose height greater than the width (*h* > *b*) are slightly more numerous than those whose width is greater than the height (*b*> *h*), whereas in landscapes the *b*> *h are* more than six times as numerous as the *h*> *b.* There is some interest in the fact that in religious images the *h* > *b are* about twice as numerous as the *b*> *h* , indisputably, because the sky is often drawn at high altitude for presentation, while in the mythological images, conversely, the width is preferred the *b* > *h being* almost twice as high (609 vs. 350) as the *h* > *b.*

The average size is from the values A_1 or G_1, the average fluctuation from the bez. A_1 applicable η can be seen. The comparison of η and A_1 in particular shows that with the average size also the average fluctuation increases, so that the relative fluctuation η **:** A_1 does not show very great differences according to class and department.

In order to take into account not only the average fluctuation but also the extreme fluctuation, I give in the following table the extremes E' and $E_{,}$ as well as the difference $U' - U_{,} = (E' - A_1) - (A_{1,} - E_{,})$. The values given also E'' and $E_{''}$ represent the extremes E' and $E_{,}$ before the immediately preceding and following values of the distribution panel.

IV. The extreme values and the extreme variation for genre, landscape, still life, religious and mythological.

E = 1 cm.

			e'	e''	E''	$E_{,}$	$U' - U_{,}$
Genre	$h > b$	H	223	215	13	12	+ 126
		b	212	162	10	9	+ 134
	$b > h$	H	273	240	12	11	+ 156
		b	401	351	16	16	+ 243

Landscape . ,	$h > b$	H	300	269	16	14	+ 138
		b	244	240	16	11	+ 117
	$b > h$	H	340	340	7	7	+ 218
		b	464	464	10	10	+ 293
Still life. ,	$h > b$	H	241	238	22	22	+ 102
		b	228	190	16	16	+ 120
	$b > h$	H	221	204	17	16	+ 95
		b	343	317	20	19	+ 172
Religious . ,	$h > b$	H	1000	610	13	10	+ 739
		b	769	568	8th	7	+ 562
	$b > h$	H	666	595	11	11	+ 454
		b	1277	1000	17	17	+ 982
Mythological.	$h> b$	H	411	411	21	21	+ 149
		b	325	324	16	14	+ 131
	$b > h$	H	290	222	14	14	+ 70
		b	510	485	20	17	+ 211

So z. For example, the greatest height h , which occurred in a gen,9mm $h > b$, 223 cm, the next largest 215 cm; the smallest 12 cm, the next smallest 13 cm; usf The absolute highest height and width has occurred in religious images. The comparison of the values E' and E'' on the one hand, E , and E'' on the other hand, it can be seen that, in general, the parts of the primary distribution boards terminating with the largest values show larger irregularities than those starting with the smallest values; only the landscapes and the mythological do not seem to confirm this, but even in these two classes the addition of the still adjacent values would make the difference indicated between the top and the bottom of the panel stand out.

The u values of Table III are most useful in assessing asymmetry . According to them, the asymmetry is bez. A everywhere negative and strong. Also, on the basis of those values, it can be noted that h agrees with the corresponding b in the asymmetry, by considering the small differences which the table between them shows to be random. Only with the religious is the difference in this relationship a little bigger; but the great irregularities of this class do not allow at all to gain secure legal provisions from it.

The values $U' - U$, Table IV confirm the existence of substantial asymmetry and at the same time prove the inverse law for the asymmetry of $u = \mu' - \mu$, and $U' - U$, inwhich both series of values have opposite signs.

Incidentally, even the wide divergence of the values *A* and *C* in Table III, as well as the position of *C* below *A,* reveal the presence of strong asymmetry from the negative direction. The comparison of *G* with *C* further teaches that the asymmetry of bez. *G* far less and for still lives *h* > *b* even from opposite direction than bez. *A* is. This is related to the fact that *G is*necessarily smaller than *A* and, since *C is also* smaller than *A* , above or below *C*but at least the latter is closer than *A*

§ 174. [In order to prove the logarithmic law of distribution on the dimensions of the gallery paintings, the arithmetically reduced intervals of Plate II must be reduced to logarithmically reduced. For this purpose, by means of the information on extreme values given in Table IV, the total area within which the observed measures are moving, and in particular the range of the interval to which the measures referred to as "remainder" are divided, and then the distribution to calculate the arithmetically reduced measures on the logarithmic intervals by interpolation.]

[As examples, I choose: genre *h* ; *h*> *b* and *h* ; komb., furthermore landscape *h* ; *b* > *h* and still life *b* ; *h* > *b* and thus obtain the following comparison table between theory and experience, in which the logarithmic interval equal to 0.08 with the lowest limit 0.76 = log 5.8 was assumed. Immediately, the elements of the four sample tables are listed.]

V. Logarithmically reduced distribution panel for genre, landscape and still life.

$i = 0.08.$

a	genre				landscape		still life	
	h ; *h* > b		*h* ; comb.		*h* ; *b* > *h*		*b* ; *h*> *b*	
	emp.	theor.	emp.	theor.	emp.	theor.	Emp.	theor.
0.80					-	0.5		
0.88					3	1		
0.96	-	1	-	2	4	3		
1.04	6	2	11	4	13	6	1	-
1.12	8th	6	14	10.5	17	14	1	0.5
1.20	9	14	16	24	19	27	1	1
1.28	20	28	34	47.5	35	49	3	3
1.36	56	49	94	82	84	81	7	7
1.44	68	73	114	123	104	119	9	14
1.52	98	94	164	161	170	159	27	23
1.60	107	103	190	183	198	192.5	33	34

1.68	99	99	191	184	217	210	41	43
1.76	79	88	159	170	216	210	52	49
1.84	76	72	145	145.5	196	192.5	50	48
1.92	61	55	110	115.5	147	163	37	39
2.00	30	38	75	85	148	128	27	25
2.08	26	24	78	58	89	93	10	13
2.16	27	14	56	37	68	62	6	6
2.24	3	8th	11	22	18	38.5	2	2
2.32	2	4	9	12	14	22	1	0.5
2.40	-	2	6	6	13	12		
2.48	-	1	-	3	11	6		
2.56			-	2	10	3		
2.64					-	2		
m =	775	775	1477	1477	1794	1794	308	308

VI. Elements for genre, landscape and still life after logarithmically reduced blackboard.

	genre		landscape	still life
	h ; $b > h$	h ; comb.	h ; $b > h$	b ; $h > b$
G	1.067	1,697	1,738	1,758
C	1,653	1,083	1,731	1,768
$\mathbf{D}_p$	1,605	1,634	1,712	1,796
$\mathbf{D}_i$	1.602	1,642	1,716	1,788
G	46.5 cm	49.8 cm	54.7 cm	57.3 cm
C	45.0 cm	48.2 cm	53.8 cm	58.6 cm
$\mathbf{T}_p$	40.3 cm	43.1 cm	51.5 cm	62.5 cm
$\mathbf{T}_i$	40.0 cm	43.9 cm	52.0 cm	61.4 cm
u	+ 125	+ 231	+ 112	- 36
e ,	0,160	0,170	0.201	0.176
e'	0.222	0.233	0.227	0.138
p	0.774	0.778	0.731	0.737

[The comparison between the observed and calculated values shows that the four K.-G. in proportion to the number *m of* the underlying specimens, prove the logarithmic law of distribution fairly uniformly. In particular, it can be noted that the combined measures of the height of genre, as well as the other departments, conform to the demands of theory; as in the example table of chap. XXI the dimensions for h> b and *b> h were* not divorced. Note, moreover, that there with the small number *m* =If a satisfactory proof of the theory has been obtained, it would seem more correct to be cautious in the formation of classes and divisions of the paintings, than to allow an exceedingly large number of copies to remove the illegality caused by the inadequacy of the classification expect. With regard to the elements, it should be emphasized that the empirically and theoretically determined densest values D_i, and D_p are little different, but that the ratios *p are* consistently below the theoretical limit ¼ π . The asymmetry is for still lifes bez. ***D*** negative, thus bez. ***G*** **-**or, as already noted above, bez. *G* - positive.]

§ 175. Finally, the following information about the measures for the ratio of height and width and for the area of galleried images of interest.

In Chap. It has been shown in XXII that, in determining mean ratios, essentially only the summary or geometric mean comes into consideration. If we now hold on to the geometrical means of *h* **:** *b* or *b : h* to be obtained divisorically from Table III , by prefixing *h : b* for *h*> *b* and *b* **:** *h* for *b*> *h* , we find following table:

VII. Geometric means and the relations of height and width.

	$h : b$ $h > b$	$b : h$ $b > h$	$b : h$ comb.
Genre	1.25	1.34	1.02
Landscape .	1.25	1.38	1.28
Still life. ,	1.26	1.39	0.99

These determinations contain what seems to me to be the very interesting result that the ratio of the larger to the smaller dimension in the different image classes has the same value (very different from the golden ratio) - for the differences in the table may be considered random - but a different one , depending on $h > b$ or $b > h$. For $h > b$ the height to the width behaves noticeably exactly like 5 **:** 4, for $b > h$ the width to the height is about 4 **:** 3.

Furthermore, it may be remarked that while in the two sections $h > b$ and $b > h$ the height deviates from the width in such considerable proportions, the ratio of the two in the combined sections in genre and still-life is almost equal (the Values 1) are accommodated. However, one might think, since *h* from *b* to a lesser ratios at *h*> *b* than *b*> *h* is different, the latter would give the combination to swing to his side; but this is roughly compensated by the fact that both genre and still life are *h* > *b* in greater numbers enter into the combination than the *b*> *h*. In contrast landscapes where *b* > *h* tremendously outweigh in number, such compensation does not occur.

For genre I have the geometric mean of *h* **:** *b* for *h* > *b* and *b : h* for *b* > *h*followed by special directions. The constancy of these relationships seems all the stranger when examined in particular for pictures of different galleries, by finding so nearly the same values that the deviation can be regarded as accidental, if only each gallery or combination of galleries presents a sufficient number of such pictures so as not to give too much scope to the uncertainty of the provision. This is evidenced by the following table, in which the specimens of such galleries, which presented only a small number of genre pictures, are taken together for the purposes of remedying.

VIII. Geometric means of *h* : *b* and *b* : *h* in genre pictures of various galleries.

	h > *b*		*b* > *h*	
	m		*m*	
Dresden	151	1.28	119	1.33
Munich a) and b); Frankfurt	126	1.25	103	1.33
Petersburg	122	1.24	87	1.34
Berlin a) and b)	74	1.22	60	1.36
Paris	62	1.23	82	1.36
Brunswick and Darmstadt	57	1.24	58	1.32
Amsterdam and Antwerp	48	1.24	24	1.33
Vienna, Madrid, London. , , ,	48	1.30	97	1.37
Leipzig a) and b)	48	1.29	34	1.32
Brussels, Dijon, Venice, Milan, Florence	39	1.23	38	1.35
	775		702	

Even with the absolute value of the width b , the relationship between h and b does not seem to change significantly after the examination of genre pictures . For I find the following geometric means of the following numbers m of specimens between the following size limits:

IX. Geometric means of h : b and b : h at different size of b (for genre).

Intervals of b	$h > b$		h	
	m		m	
0 - 29.5	274	1.27	42	1.32
29.5 - 49.5	271	1.23	158	1.29
49.5 - 69.5	123	1.23	164	1.32
69.5 - 89.5	54	1.23	98	1.36
89.5 - 109.5	28	1.28	63	1.37
Rest	25	1.23	177	1.39

For the geometric mean of the surface areas *hb the* following values are obtained in qcm.

X. Geometric mean of *hb*.

E = 1 sq. Cm.

	$h > b$	h	comb.
Genre. , ,	1747	3874	2550
Landscape . ,	4303	4098	4128
Still life. , ,	4189	5018	4496

The arithmetic mean of the *hb* I have determined because of the great difficulty of its determination only for genre $h > b$ and found 3289 qcm, which, as you can see, greatly deviates from the geometric mean.

Out of the total 10558 pictures which have been entered in Table II, the three largest in space are three pictures of PAUL VERONESE, all representing the feasts in which Christ was present, namely:

Banquet at Levi (Luc V) *h* = 595 cm *b* = 1277 cm (Venice, No. 547)

Wedding at Cana *h* = 666 cm *b* = 990 cm (Paris; - 103)

Banquet at the Pharisee *h* = 515 cm *b* = 1000 cm (Venice, - 513).

The three smallest pictures are three landscapes on copper, two of the same size allegedly by PAUL BRILL: *h* = 7.4 cm, *b* = 9.1 cm (older Pinakothek to Munich, 2nd Dept. 244 a u c) and one of JAN BREUGHEL: *h* = 7,4 cm, *b* = 9,9 cm (Milan No. 443); according to which the surface area varies between 67.34 and 759815 qcm or the largest picture can take 11283 times the smallest picture.

Square images were only 84 di 1 in 126 among the 10558 images taken for examination.

XXVII. Collective items from the field of meteorology.

§ 176. [The daily rain heights for Geneva. - PLANTAMOUR has already given an analysis of the Geneva rain conditions in his "Nouvelles études sur le climat de Genève" in the section "de la pluie" [1], He bases himself on the fifty-year observations of rainy heights and rainy days during the years 1826-1875. But since he bases his calculations only on monthly values for the frequency and amount of rain, and his goal is the regular distribution of rain during the year, and the nature of the individual months of the year in terms of dryness or humidity, the following examination can not be conducted in the style of PLANTAMOUR's. For here it is the proof of asymmetry and the probation of the logarithmic distribution law for the rain heights, for which the 50-year monthly values by no means sufficient for the very large fluctuations between the individual values. Rather, it must go back to the daily rains.]

[1] [Published in: Mémoires de la société de physique et d'histoire naturelle de Genève. Tome XXIV; II. Lot. Genève 1875-76. Pp. 397-658.]

[The research material can be found in the archives of the sciences physiques et naturelles of the Bibliothèque universelle de Geneve under the monthly given meteorological tables. There, for every rainy day, the rainfall in millimeters, down to a tenth of a millimeter, under the heading: "Eau tombée dans les 24 heures" recorded. No account is taken of the form of precipitation, rain or snow [2], However, I chose not the period treated by PLANTAMOUR, but the series of the 48 years from 1845-1892. For from 1846 on a new apparatus was used, and at the same time there was a more careful determination of the level of rainfall, immediately after the cessation of rainfall, instead of as usual until one day of the last observation in the evening, in practice. [3)]]

2) [PLANTAMOUR says, op. Cit., P. 627): Les chutes de neige sont en général très-peu abondantes à Genève, et la neige ne recouvre ordinaire le sol que pendant un petit nombre de jours, rarement plus de quinze jours.]

3) [In this regard, plantamour aa 0. (p. 627) makes the following statement: A partir de l'année 1846 on s'est servi d'un nouveau appareil, dont l'entonnoir avait un diamètre beaucoup plus considérable, 37 centimètres, le vase de jauge est une éprouvette graduate of the capacité d 'un literé, portant 100 divisions, ce qui correspond à un chute d'eau de 10 millimètres, chaque division correspondant ainsi à un dixième de millimètre; de plus, on avait le soin de recueillir et de mesurer l'eau immédiatement après que la pluie avait cessé.]

[The appearance of the primary distribution panels is shown in the following sample, which indicates the beginning, middle part and end of observed values for the month of January:

I. Sample from the primary distribution panel for the rainy seasons of January .

$m = 477$; $i = 0.1$ mm.

a	z	*a*	z	*a*	z	*a*	z
mm		mm		mm		mm	
0.0	16	5.0	3	6.1	6	19.6	1
0.1	9	5.1	2	6.2	2	19.7	1
0.2	18	5.2	2	6.3	5	19.8	1
0.3	19	5.3	5	6.4	5	21.4	1
0.4	9	5.4	1	6.5	1	21.6	1
0.5	10	5.5	2	6.6	1	21.8	1
0.6	11	5.6	4	6.7	2	23.6	2
0.7	18	5.7	5	6.8	1	28.4	1
0.8	8th	5.8	1	6.9	1	30.4	1
0.9	10	5.9	4	7.0	2	32.7	1
1.0	10	6.0	1	7.1	4	40.0	1

In fact, all months show the strongest accumulation in the interval 0 - 1 mm, but already from 2 mm onward one finds a rapid decrease in the values which form after long undecided swaying very irregular end sections with scattered a . The length of the latter, however, varies greatly for each month, closing at 31.3 mm for February and only 97.6 mm for October, whereas its beginning for that month is about 12 mm for this month to be set to 18 mm. For January, the limits of this final section are 12mm and 40mm.]

(This general information already indicates the existence of a very strong asymmetry for all months of the year, which, at the same time as the major values during the year, appears in full clarity in the following table of elements:

II. Rainfall elements for each month of the year according to primary distribution tables.

E = 1 mm

	Jan.	Febr.	March	April	May	June	July	Aug.	Sept.	October	nov.	December
M	477	437	532	621	637	596	521	531	497	617	572	505
A_1	4.45	4.17	4.60	4.94	6.12	6.58	6.95	7.93	8.46	8.49	6.09	4.97
C_1	2.5	2.1	2.6	3.0	3.6	3.3	3.8	4.1	4.6	4.9	3.3	3.0
H	3.82	3.79	4.03	4.14	5.24	5.93	6.11	7.10	7.57	7.49	5.23	4.11
$\eta : A_1$	0.86	0.91	0.88	0.84	0.86	0.90	0.88	0.90	0.89	0.88	0.86	0.83
e'	40.0	31.3	51.0	38.3	80.7	82.5	60.6	61.1	82.6	97.6	56.7	40.0
$U'-U_,$	+31.1	+23.0	+41.8	+28.4	+68.5	+69.3	+46.7	+45.2	+65.7	+80.6	+44.5	+30.1
u	-131	-167	-164	-197	-195	-196	-177	-189	-177	-209	-168	-141
$u : m$	0.27	0.38	0.31	0.32	0.31	0.33	0.34	0.36	0.36	0.34	0.29	0.28

The values of the lower extremes $E_,$ have not been included here, since they are equal to 0.0 mm throughout. They are everywhere, as the above sample shows, in multiple edition.]

[The divergence of the values of A and C by 2 to 4 mm on the one hand, the differences $U' - U_, = (E' - A) - (A - E_,)$ on the other hand, and in particular the differences $u = \mu' - \mu_,$ prove coincidentally the presence of significant asymmetry. A_1 for all months of the year. The same is everywhere negative, according to the sign of u , and shows no great fluctuations in size; because the relative values of u bez. m , di$u : m$, are almost constant, and their small differences do not betray a legitimate gait, so they should be considered as coincidental.]

[Furthermore, the course of m , A and η in the above table deserve attention. From the m following values that the frequency of rain has two periods during the year whose minima the months of February and July, and their maxima are the months of May and October, while in between a constant rise or fall occurs. Only September breaks the regularity; however, this disturbance is to be regarded as accidental, since it lacks the *m*- values of the years 1826-1875 to be taken from PLANTAMOUR's table [4)], for which the month of January then interferes. This is from the following comparative compilation of *m* Values for the periods 1826-1875 and 1845-1892, where the order of the values from left to right corresponds to the order of the months from January to December:

1836-1875	505	413	496	525	589	532	471	503	521	576	539	454
1845-1892	477	437	532	621	637	596	521	531	497	617	572	505

Contrary to the *m* , the *A* show only one period, which runs without disturbance and has its minimum in February, its maximum in October. Parallel to this, the values of the η , ιε the mean deviations, are approx. *A* , whose minimum also falls to February, while reaching its maximum one month earlier, in September. The large values of η , which are very close to the *A* itself, reveal the magnitude of the fluctuations that take place between the different levels of rain. The relative mean variation is, as the values η **:** *A* indicate, approximately constant, equal to 0.9.]

[4)] A. a OS 628.

[According to this, the average height of the rain increases during the year from February to October, before falling again until February. A correct picture of the distribution of rain on the individual months is not obtained in this way. Because here is also the frequency of rainfall into consideration. If, accordingly, the total amount of rain occurring in one month during the 48-year period is not allocated to the individual rainy days that have actually taken place, but to all the days at all, we also get the rainfall, as well as the frequency of rainfall Regens, within the year a twofold periodicity, as demonstrated by PLANTAMOUR. One finds for the individual months of the year the following average rainfall for each day of the month,

1826-1875	1.57	1.29	1.52	1.89	2.55	2.53	2.29	2.59	3.14	3.26	2.47	1.65
1845-1892	1.42	1.34	1.64	2.13	2.62	2.72	2.43	2.83	2.92	3.52	2.42	1.68

In fact, here the two minima coincide with the months of February and July; the first maximum fluctuates between May and June, while the second maximum belongs to October ([5)] .]

[5)] [With regard to this twofold periodicity, PLANTAMOUR, loc. Cit., P. 640): "Cette division de l'année en deux seasons humides et seasons sèches, l'une de celles-ci tombant sur l 'été, accuse très nettement l'influence du climat méditerranéen; en effet, le caractère du climat méditerranéen est la sècheresse de l'été, tandis que dans les autres régions de l'Europe continental, l'été n'est pas une saison sèche. "]

[In order to prove the logarithmic distribution law at the rain heights, I choose the four months January, April, July and October, which allow a complete insight into the occurring conditions. The logarithmically reduced distribution board, as well as the arithmetically reduced primary board, are used as a basis. But at the transition to the logarithmic intervals, the values should be 0.0 mm, to which the logarithmic value - ∞If it were not to disappear from the blackboard, a determination must be made as to the perception of the rainy days recorded with these values. Since this level of rainfall is apparently intended to indicate a real but negligible precipitation of less than 0.1 mm in height, it seems justified to place 0.05 mm instead of 0.0. To alleviate this arbitrariness, log 0.05 = - 1.3 is chosen as the limit of the first and second logarithmic intervals, so that one half of each of these values falls within the first occurring interval, the other half into the next. The size of the logarithmic intervals was further set equal to 0.2. Thus, the *a* values vary between the limits 0 and 100 mm, the logarithmic ***a*** values between the limits -1.5 and +2.1, as shown in the following distribution tables. In the logarithmic table are at the same time given the theoretical values as given by the law. Immediately afterwards the elements are listed:

III. Arithmetically reduced table of the rain heights for Geneva during the months of January, April, July, October 1845-1892.

intervals	January	April	July	October
mm				
0 - 1	133	164.5	112.5	125
1 - 2	88	81	78.5	72.5
2 - 3	43.5	65	31	60
3 - 4	28	49.5	48	31
4 - 5	27	51	28	24.5
5 - 6	28	20.5	28.5	39
6 - 7	27.5	37.5	23	26
7 - 8	14.5	25	23.5	19.5
8 - 9	16	22	15.5	26.5
9 - 10	11.5	15.5	11.5	14
10 - 11	12	16	13	21

11 - 12	10	15	14	12.5
12 - 13	6.5	9	10	14.5
13 - 14	5.5	8.5	8th	10.5
14 - 15	3	3.5	9	11.5
15 - 16	3	5.5	5	13
16 - 17	2	3.5	3.5	8.5
17 - 18	5	3.5	5.5	9
18-19	1	4	3	4.5
19-20	3	3	7	6.5
20 - 25	5	6	17	22
25 - 30	1	8th	12	17.5
30 - 40	2.5	4	9	17
40 - 50	0.5	-	3	2
50 - 70	-	-	2	6
70 - 100	-	-	-	3
m =	477	621	521	617

IV. Logarithmically reduced table of rain heights for Geneva during the months of January, April, July, October 1845-1892.

$i = 0.2$

a	January		April		July		October	
	emp.	theor.	emp.	Theor.	emp.	theor.	Emp.	theor.
-	-	5	-	2	-	1	-	3
- 1.4	8th	4	10	2	7	2	1	3
- 1,2	8th	6	10	5	4	4	1	5
- 1.0	9	9	17	8th	12	7	17	7
- 0.8	9	14	10.5	13	9	11	10.5	11
- 0.6	28	19	30.5	21	20	16	23.5	17
- 0,4	14	26	18.5	31	11.5	23	22.5	24
- 0.2	34	34	33.5	42.5	28.5	31	22.5	32
0	45	42	62	55.5	50	39	47	42

+ 0.2	66	50	53.5	68	52	49	52.5	51
+ 0.4	47	56	72.5	78	38	57	65.5	61
+ 0.6	53	60	**95**	85	**72**	63	52	69
+ 0.8	**67**	63	80	85	68	66	80	74
+ 1.0	53	52	74	67	64	64 [6)]	**82**	77
+ 1,2	27	27	36	38	45	47	72	69
+ 1.4	7	8th	14	15	31	26	42	44
+ 1.6	2	2	4	4	10	11	17	20
+ 1.8			-	1	2	3	6	6.5
+ 2.0					-	1	3	1.5
m =	477	477	621	621	521	521	617	617

[6)] If less values fall on the theoretically closest interval 0,9-1,1, which includes the densest value D_p, than on the preceding one, this is not due to a mistake, but to the summary of the theoretical values in the predetermined intervals. If both intervals are separated into four equal sub-intervals of size 0.05, then instead of 66 and 64, we obtain:

| 16.2; 16.3; 16.6; 16.6 | and | 16.7; 16.4; 15.6; 14.9 | .

so that in fact the maximum 16.7 falls on the subinterval 0.9-0.95, which is affected by D_p .]

V. Elements of the rain heights after logarithmically reduced board.

	January	April	July	October
G	0.313	0.387	0.484	0.563
C	0.374	0.479	0.588	0.675
$\boldsymbol{D}_p$	0.843	0,762	0.901	1,046
$\boldsymbol{D}_i$	0,800	0,620	0.679	0.933
G	2.06 mm	2.44 mm	3.05 mm	3.66 mm
C	2.37 mm	3.02 mm	3.87 mm	4.73 mm

T_p	6,97 mm	5.78 mm	7,97 mm	11.1 mm
T_i	6,31 mm	4.17 mm	4.77 mm	8,58 mm
u	- 261	- 255	- 218	- 293
e ,	0.749	0,645	0.707	0,750
e'	0.219	0,270	0,290	0,267
p	0.885	0.755	0.751	0,772

Corresponding to the strong irregularities of the empirical values, considerable differences sometimes appear between the empirical and theoretical values, which, however, are softened when taking adjacent intervals. These are therefore to be regarded as immaterial perturbations, so that the theoretical values represent an equalization of the accidents which adhere to the empirical values. It is noteworthy with respect to the elements that *G* below *C* and thus, with respect to Table II, *C* between *G* and *A* is located. This also proves the very large fluctuation of the rain heights. This is further related to the fact that the *u*- values bez. D_p as well as the *u*- values bez. A_1 are negative. The relative value of the asymmetry bez. D_p , di $u : m$, is again fairly constant, averaging 0.46 on average.]

§ 177. [The barometer deviations from the standard for Utrecht. - The asymmetry of the barometric deviations is known. QUETELET says in this regard [7] : "On a reconnu, depuis longtemps, que l'abaisement du mercure au-dessous de moyenne est en général plus grande que élévation au-dessus de ce terme." It is henceforth positive asymmetry on *A* consistently or at least in the majority of cases.To test this and at the same time to prove the bilateral GG at the barometric deviations, I take from the Dutch Yearbook for Meteorology [8]the deviation values from the monthly normal level given in the section "Thermo-Barometer-arwijkingen" for the observatory "Utrecht" and the observation time "2 o'clock in the afternoon, during the ten-year period from 1884 to 1893. However, I do not give these values to all Months, but only for January, April, July and August. I also communicate only the reduced distribution tables, as well as the elements calculated from them. It suffices to use the arithmetic treatment as a basis; because the fluctuation range of the deviation values is not so great that the effort of the logarithmic treatment would be worthwhile. Therefore, the theoretical comparison values included in the empirical values were derived from the arithmetic two-sided distribution law.i = 3 mm instead of the primary i = 0.1 mm was caused by the extreme fluctuation of January. For consistency, this interval was maintained for the other three months. It should also be noted that in the Dutch yearbook 31 January (as well as 1 March) is added to February, which explains the total number of 300 instead of 310 observed values for January.]

7) [Lettres sur la théorie des probabilités, p. 168.] For this it is of interest to compare BRAVAIS's epistolary statements, communicated by QUETELET in the attached notes, on various forms of possible laws of probabil- ity, because they show that BRAVAIS as well as QUETELET himself While recognizing the possibility of an asymmetric distribution law, it mistakenly gave the means the role of the densest value, thus failing in principle to accept the conception of the asymmetric law. The relevant passage in BRAVAIS's letter reads (aa OS 413): "On the sons and the grands écartes du barométre vers le haut de la colonne, it is still true that the two of the écarts du barométre vers] e bas; the variety que l'on aura and courgette de possibilité de la forme. dont les deux moitiés ne seront pas symmétriques; seulement l'ordonnée moyenne doit toujours partager le segment total en deux aires égales. "].

8) [Meteorological Jaarboek uitgegeven door het Kon. Nederlandsch Meteorological Instituut.]

[The results obtained are contained in the following two tables:

VI. Reduced table of barometric deviations from the standard for Utrecht, at noon 2 pm, during the months of January, April, July and October 1884 - 1893.
$\boldsymbol{E}$ = 1 mm; i = 3.

a	January		April		July		October	
	emp.	theor.	emp.	theor.	emp.	theor.	emp.	theor.
- 33	1	0.5						
- 30	1	0.5						
- 27	1	1					-	0.5
- 24	2	2					2	1
- 21	4	4	1	0.5			2	3
- 18	6	6	1	2	-	1	8th	6
- 15	9	9	6	5.5	2	3	11	12
- 12	16	13.5	16.5	14	12.5	9	23	20.5
- 9	11.5	19	22	28	20.5	21	22	30
- 6	25.5	24	42	43.5	32	39	42	38
- 3	31	30	**59**	54	63.5	58.5	42.5	41
0	31	34.5	50	53	**70**	69	34.5	40
+ 3	39.5	38	48.5	43	57	60.5	32	35

+ 6	**44.5**	39	26	29	44.5	34	30	29
+ 9	31	34	19	16	7	12	26	21
+ 12	22	24	7	7.3	1	3	27	14
+ 15	17	13	1	3			5	9
+ 18	7	5.5	1	1			3	5
+ 21	-	2					-	3
+ 23	-	0.5					-	2
m =	300	300	300	300	310	310	310	310

VII. Elements of barometric deviations.

$\boldsymbol{E}$ = 1 mm.

	January	April	July	October
normal status	760.16	759.64	760.62	759.01
A_2	+ 1.01	- 1.22	- 0.76	- 0.93
C_2	+ 2.34	- 1.35	- 0.45	- 1.28
D_p	+ 6.06	- 1.82	+ 0.71	- 2.60
D_i	+ 5.31	- 2.54	- 0.45	- 4,32
η 9)	7.72	5.15	4.05	7.15
e ,	9.86	4.86	4.93	6.31
e'	4.81	5.47	3.46	7.98
u	+ 32	- 5	+15	- 7
$\mathbf{u}$	- 103	+ 18	- 54	+ 36
p	0.737	0.783	0.789	0,790

Here, the presence of substantial asymmetry at the same time as the validity of the two-sided GG is shown on the one hand by the agreement of the empirical and theoretical values and on the other hand by the position of the principal values A , C , D_p , D_i at the ratio values p and the values of u and $\mathbf{u}$. At the same time it is evident that the succession of succession, whose existence in XXIII. Cape. In particular, for the barometric deviations of January has been numerically proven, the probation of the distribution laws in any case not impossible. However, the values of u and $\mathbf{u}$ teachcoincidentally, that the asymmetry is by no means constant during the year. Rather, a lawful course reveals itself in the course of the year, according to which the strong asymmetry of the winter and the less strong of the summer is interrupted by a vanishing or turning to the opposite in the spring and autumn. It should be noted, however, that the four months are not enough to get a complete picture for the whole year with certainty. Nevertheless, it will be concluded that the asymmetry is greatest during the winter months and shows at least the tendency to fluctuate during the year. - The mean ηindicate a legal process according to which the deviations from the normal state - as the appearance of the distribution boards already shows - are the strongest in winter and the weakest in summer. The course of the normal state itself, which was obtained as a means of many years of observations, shows the following compilation:

month	**January**	**February**	**March**	**April**	**May**	**June**
normal status	760.16	760.62	760.61	759.64	760.09	760.78
month	**July**	**August**	**September**	**October**	**November**	**December**
normal status	760.62	760.42	760.71	759.01	759.30	760.34

Thus, in January, the normal level comes very close to the annual average of 760.19; in April and October it is smaller, in July it is larger than the annual average.]

9) [The values of η were calculated as averages of deviations from normal , regardless of the A_2 and the resulting small deviation of the ten-year mean from the normal.]

§ 178. [The thermometer deviations from the standard for Utrecht. - In a similar way, as it happened for the barometric deviations, the asymmetry should now also be investigated for the deviations of the thermometer from the normal state, and the validity of the bilateral GG should be proved with arithmetic treatment. For this purpose, the deviation from the long-term average for Utrecht during the years 1884-1893, in the afternoon at 2 o'clock, during the months of January, April, July and October is taken from the Dutch Yearbook of Meteorology. The values are given in degrees of the 100-part scale, down to tenths of a degree. However, for the course of a month they do not refer to the mean of the whole month as the barometric deviations, but in order to take into account the livelier pace of the mean temperature, to the normal values of the first, second and third decade of each month. The rise and fall of the latter during the year shows the following composition:

month		January	February	March	April	May	June
normal status	1st decade	+ 2 °, 78	3 °, 97	6 °, 56	9 °, 88	15 °, 15	18 °, 97
	2nd »	+ 2 °, 73	4 °, 95	7 °, 43	12 °, 46	16 °, 15	19 °, 86
	3. »	+ 3 °, 30	5 °, 94	8 °, 45	14 °, 26	17 °, 25	20 °, 37

month		July	August	Septbr.	Oktbr.	Novbr.	Dezbr.
normal status	1st decade	+ 2o °, 86	21 °, 28	19 °, 05	15 °, 52	8 °, 65	4 °, 71
	2nd »	+ 21 °, 30	20 °, 94	18 °, 07	13 °, 22	6 °, 82	3 °, 82
	3. »	+ 21 °, 50	20 °, 32	17 °, 13	10 °, 94	5 °, 72	3 °, 23

According to this, the average normal level for January, April, July and October is in order: 2 °, 94; 12 °, 20; 21 °, 22 and 13 °, 23.]

[If you now determine the size of the reduced interval equal to 1 °, you get the following results:

VIII. Reduced panel of thermometer deviations from the standard for Utrecht, in the afternoon at 2 o'clock, during the months of January, April; July, October 1884-1893.

E = 1 ° Celsius; i = 1.

a	January		April		July		October	
	emp.	Theor.	emp.	theor.	Emp.	theor.	emp.	theor.
- 12	-	1						
- 11	-	1.5						
- 10	2.5	2.5	-	1	1	-		
- 9	4.5	4	2	2.5	1	1	2	0.5
- 8th	3.5	6	2	5	1	3	1	1.5
- 7	10	8th	11.5	9.5	7.5	7	2	4
- 6	13.5	11	21.5	15	6	13	12.5	11
- 5	18	15	25	22	21	21	20	21
- 4	20.5	19	15.5	26	31.5	29	26.5	32
- 3	26	22.5	**37.5**	28	38	34	**45.5**	40
- 2	22.5	26	28	28	**48**	36	41.5	41
- 1	23.5	28	32	26	38	34	33	38
0	**31**	30	18	24.5	25	31	42	34
+ 1	25.5	30	17.5	22	14.5	27	27	27
+ 2	32.5	27.5	15	19.5	27	22	24.5	21

+ 3	22.5	23	12	16.5	10.5	17	9.5	15
+ 4	15	17.5	16.5	14	11.5	12.5	5	10
+ 5	14	12	10	11	7	8.5	10	6
+ 6	8.5	7.5	12.5	9	8.5	6	3.5	4
+ 7	4	4.5	5.5	6	4	4	1.5	2
+ 8	1.5	2	6.5	5	5	2	3	1
+ 9	1	1	4.5	3	1.5	1	-	1
+ 10	-	0.5	2	2	2	1		
+ 11			3	2	0.5	-		
+ 12			2	1				
+ 13			-	1				
+ 14			-	0.5				
m=	300	300	300	300	310	310	310	310

IX. Elements of the thermometer deviations.

$\boldsymbol{E}$ = 1 ° Celsius.

	January	April	July	October
av. normal status	+ 2.94	+ 13,20	+21.22	+13.23
A_2	- 0.58	- 0.50	- 0.89	- 1,11
C_2	- 0.32	- 1.28	- 1.50	- 1.38
D_p	+ 0.61	- 3,11	- 2.37	- 2.49
D_i	+ 0.08	- 2,80	- 2.00	- 2.67
η [10)]	3.17	3.71	3.08	2.59
e ,	3.76	2.09	2.01	1.68
e'	2.57	4.70	3.49	3.06
u	+ 19	- 50	- 46	- 18
$\boldsymbol{u}$	- 57	+ 115	+ 84	+ 91
p	0.782	0.701	0.588	0.804

Here, too, the correspondence between theory and experience is satisfactory, although, in accordance with the relatively smaller reduction step, apparently less well than for the barometric deviations. The asymmetry is positive only for January bez. *A* ; negative for the other three months. That exception could now be considered random, since the observed *u*-value is small. However, as the same direction of asymmetry, again with similarly weak values as in January, was found in December, which I used for comparison in this respect, it may be assumed that the asymmetry is negative during the greater part of the year bez. *A*is approaching zero during the winter, with a tendency to turn positive. Finally, it is worth mentioning that the average fluctuation η ισ fairly constant for the months studied (and probably for the whole year).]

10) [The η refer here, as with the barometric deviations, to the normal level.]

§ 179. [The daily variations of the temperature for Utrecht. - While the thermometer deviations refer to a certain hour of the day (2 o'clock in the afternoon), the daily variations indicate the differences between maximum and minimum of the daytime temperatures. Their collective treatment according to the arithmetic principle has a double interest because of the remarks in § 21. Because they can be considered free from succession dependence and thus allow an unhindered probation of the distribution laws. They were also used by QUETELET as a basis for the discussion of asymmetry; therefore, the comparison between the treatment of these K.-G. after two-sided GG and the presentation of QUETELET's in the "Lettres sur la théorie des probabilités" a direct insight,

[First, I share the results obtained in the following two tables. The study material was taken from the Dutch Yearbook for the period 1884-1893 and the observatory Utrecht with the exception of the months January, April, July and October, as for the barometric and thermometric deviations. It can be found there in the section "driemaaldaagsche Waarnemingen" under the heading "Temperatuur". As a reduced interval, 1 ° Celsius was chosen (as in the corresponding distribution boards given by Brussels for Quéneté):

X. Reduced table of daily variations of temperature for Utrecht during the months of January, April, July, October 1884-1893.

E = 1 ° Celsius; *i* = 1.

a	January		April		July		October	
	emp.	theor.	emp.	theor.	emp.	Theor.	emp.	theor.
-	-	1						
0.5	3.5	5	-	2	1	-	-	1

1.5	22.5	22	4	4	0	0.5	6	5
2.5	49	48	5.5	8th	2.5	2	21	18.5
3.5	**62**	59	18.5	16	8th	8.5	32.5	41
4.5	51	53	33.5	25	18.5	24	**65.5**	58
5.5	48	43	29.5	34	47.5	43	54	57
6.5	29.5	31	38	40	55	54	48	48
7.5	16.5	19	**38.5**	40	**56.5**	52	37.5	35
8.5	7.5	11	37	36	43	44	25.5	23
9.5	4.5	5	31	30	29	33	8.5	13
10.5	4	2	17	23	21.5	22.5	7	6
11.5	0	1	24.5	17	15	13.5	4.5	3
12.5	0	-	11	11	4.5	7	-	1.5
13.5	2	-	10	7	5	3.5		
14.5			1	4	2	1.5		
15.5			0	2	1	1		
16.5			1	1				
m =	300	300	300	300	310	310	310	310

XI. Elements of the daily variations of the temperature.

E =1 ° Celsius.

	January	April	July	October
A_2	4.53	7.69	7.64	5.75
C_2	4.26	7.55	7.40	5.56
D_p	3.24	6.87	6.59	4.73
D_i	3.54	7.25	7.10	4.74
e ,	0.97	1.95	1.28	1.15
e'	2.26	2.77	2.33	2.17
u	- 28	- 11	- 27	- 21
u	+ 120	+ 52	+ 90	+ 95
p	0.791	0.829	0.771	0.814

Based on these results, the validity of the bilateral GG can not be doubted. The differences between the empirical and theoretical values are on average lower than in the corresponding comparison tables of the barometer and thermometer deviations. Likewise, the principal values and the ratios of the *p* satisfy the theoretical requirements, while at the same time the asymmetry is determined partly by the constancy of their direction, partly by their particularity in the *u*Values of January's emergent strength as essential documented. Thus, while the daily variations on the whole yield more favorable results than the barometric and thermometric deviations, both of which involve succession dependence, the absence of succession-dependentness in fact seems to favor the development of the laws of pure chance.]

[To further compare QUETELET's discussion of asymmetry [11] , the following must be said about the method of its investigation. QUETELET assumes that, given essential symmetry, the W. positive and negative deviations from the arithmetic mean are the same, and concludes that the asymmetry in W.'s inequality is due to the mutual deviations from the mean. He thus illustrates the likelihood ratios occurring here through the urn, which contains an infinite number of black and white spheres in different but definitely to be chosen ratios. In particular, he is a tabular summary of the W . which consist in pulling 16 balls for the appearance of balls of one kind when 50; 55; 60; .... 90; 95 balls of one kind under each 100 balls occur. With these tables of theoretical W. he compares now the tables of empirical W. resulting from the reduced distribution tables for the daily variations of temperatures (for Brussels) by dividing the *z of* each interval by the corresponding *m* . Thus, for the month of January, which he bases on his explanations, he finds that the course of empirical studies is considerably closer to the course of those theoretical words for which the numbers of white and black spheres are 80:20, and notes that the analogy would be even greater if the ratio 80 **:**20 through 81 **:** would be replaced 19th From this he concludes with regard to the previously provided by it mean following [12)] : "1) il existe une variation diurne de température de quatre à cinq degrés, ou plus de exactement 4 °, 7; elle est donnée par la moyenne de toutes les observations; 2) cette variation subit l'influence de causes inegales; 3) les causes qui tendent à faire tomber la variation diurne à son minimum, ont plus de chances en leur faveur que celles qui tendent à l'élever à son maximum, et les chances sont dans le rapport de 81 à 19, ou plus simplement de 4 à 1; 4) les distances de la moyenne aux deux valeurs limites sont réglées par ce même rapport de 4 à 1 «].

[11)] [Lettres sur la théorie des prob .; Lettre XXV: Des dam accidentals quand les chances sont inegales; Lettre XXVI: Loï de deux de deux événements, dont les chances sont inégales. See tables (see chapter XXV.).]

[12)] A. a. OS 181.

[From this it can be seen that QUETELET's theory is in principle inadmissible in so far as the arithmetic mean is regarded as the most probable value even in the case of predominant asymmetry. But if, nevertheless, this erroneous assumption seems to gain support from experience, it must be borne in mind that the comparison between theory and experience is based only on the appearance of the plates, ie the position of the extreme values with respect to the mean and the course of the intermediate ones Values, supports. As a result, the whole examination method has only a slight sharpness and bears the character of the incomplete. On the other hand, it should be emphasized that the conception of QUETELET's leads to the two-sided GG as soon as the densest value as defined by the law of proportionality takes the place of the arithmetic mean. The addition to the XIX. Chapter (§ 136) illustrates this connection.]

XXVIII. The asymmetry of the error series.

§ 180. [There is no doubt that the error series K.-G. representing the same treatment as the K.-G. the previous chapter. However, it is questionable whether, on the one hand, it is principally necessary, on the other hand, that experience shows that it is advantageous to apply the methods of collective asymmetry, or whether it is not the theoretical and empirical grounding of the assumption of essential symmetry. After this question has been left open in § 8, she should find her answer here. The separation of the theoretical standpoint from the empirical one is not idle. For, if the principles of asymmetry are applied in principle, their application will always carry empirical advantages if only the treatment is sufficiently sharp. to make the difference between the arithmetic mean and the densest value stand out. But it is conceivable that the two-sided GG, even if it is not required by theory, nevertheless proves itself in experience, insofar as it - compare § 95 - the empirically different$m'$and m , bez. *D*bears, whereas, according to simple GG instead of the equally empirically differentμ'andμ, bez. *A is*to be seton both sides ½*m*.]

[To solve the theoretical side of the question, which is mainly of interest, it is necessary to investigate the asymmetry of error series, for which a system of similar series of observation values of the same kind is best suited. Furthermore, any merely empirically obvious advantages will become apparent if both the two-sided and the simple GG are comparatively tested on the distribution tables of error series; In this case, one would prefer rows with a large *m* , because it is to be expected that they will develop the typical form of the error tables in the purest possible purity.]

[For one and the other purposes, the series of astronomical observation errors examined in this chapter, which are given to me by the Observator of the Observatory in Strasbourg, Dr. Kobold, at the same time with the following information about the origin of the same were communicated.]

[At the base are observations on the REPSOLD meridian circle of the observatory, which were made in the years 1884-1886 by one and the same observer. On the one hand, such an observation is intended to determine the time at which the observed star passes through the meridian and, on the other hand, determine the zenith distance in which the passage takes place. It is therefore composed of two different acts. The first act, since the transit time is electrically registered, is to press the button at the moment the star passes a vertical thread of the instrument. He can, since twenty-three such vertical threads are present, be repeated often, whereby each time the corresponding time is fixed. The second act is the exact setting of the instrument, as soon as the star approaches the middle of the 23 threads. With regard to its execution, the following is true. to notice. The device of the instrument was a deviating from the usual one in that the zenith-distance fine adjustment was not carried out (as usual) by means of a key, but mediated by a chain run around a knob located on the instrument's clamping arm and, since the clamp arm was firmer Connection with the instruments, was always in the immediate vicinity of the eyepiece was. Both acts can therefore be carried out without any mutual interference if the instrument has the position in which the clamp is located on the east side. Then the observer can hold the button in his right hand and get the fine adjustment with his left hand. If, however, the instrument is in the opposite position, there is a conflict between the two acts in so far as the adjustment at zenith distance necessitates the removal of the probe, which can be resumed only after its execution in order to register the passage time for the middle thread. As a result, a different delay occurs in different observers, so that the observation for the middle thread is disturbed by the fine adjustment in the zenith distance. The two positions of the instrument are distinguished by the terms "terminal East" and "terminal West". - It should be noted that this conflict would not occur if an observer were to be able to register with one hand as with the other,

[Of these observations, the part relating to the determination of the transit time was used to calculate the distances of the mentioned vertical filaments, ie the time required for a star in the equator to pass through the interval of two filaments. The threads were sequentially marked by the numbers 1 to 23. The distances between the middle thread 12 and the threads 2, 5, 6, 10, 14, 18, 19, 22; they are called thread distances 2 - 12; 5 - 12 and so on. Further, the observation material was divided into four groups, because on the one hand - as mentioned above - the East instrument position differs from the West position with respect to the simultaneous determination of the zenith distance, and on the other hand, apart from the majority of the night observations, there were also day observations in which other lighting conditions prevail. However, by avoiding the middle thread 12, which is the only one to be considered when disturbed by the zenith distance fine adjustment, the difference between the two layers East and West can be substantially eliminated; and in fact the same series of observations gave the distances against the thread 2 in both positions. It seemed, however, of particular interest to maintain that distinction in order to observe its possible influence on the results of the following investigation. In order to assess the relatively large errors of observation, it should also be borne in mind that the observations, because they should serve to determine the thread distances from which material extending over several years is selected so that the various conditions are as much as possible. If one had wanted to determine the average error of observation, one would have had to choose temporally close observations.]

§ 181. [The material provided consists of four groups, which are designated as follows :

α) East terminal; night observations

β) East clamp; Tagbeobachtungen

γ) West terminal; night observations

δ) West terminal; Tagbeobachtungen.

Each group contains, according to the eight thread distances, as many rows of observation values, the form of which can be seen from the following sample taken from group α). The unit of measurement used here and in the following is the time second = 1^s

I. Sample from the observation series α) East clamp; Night observations.

$\boldsymbol{E} = 1^s$

Time of observation	star	2 - 12	5 - 12	6 - 12	10 - 12	14 - 12	18-12	19 - 12	22 - 12
1884 June 24th	δ Ophiuchi	37.28	31.10	22.28	13.87	14,60	22,80	31.70	37.96
July 1	η Librae	37.34	31.14	22.39	14.07	14.61	22.87	31.70	37.92
1885 January 14	α Orionis	37.65	31.31	22.51	14.11	14.48	22.65	31,60	37.98
1886 March 35	η Booties	37.55	31.17	22.35	14.03	14.68	22.77	31.80	38.02

From these series of observations, the following elements can be obtained for the eight thread distances:

II. Elements of the thread distances .

$\boldsymbol{E} = 1^s$.

α) East terminal; Night observations.

thread distance	2 - 12	5 - 12	6 - 12	10 - 12	14 - 12	18 - 12	19 - 12	22 - 12
m	115	115	114	114	115	114	115	112

A	37.428	31.190	22.333	14.036	14.591	22.894	31.711	37.989
H	0,099	0.094	0.084	0,099	0.098	0,099	0.094	0.082
e '	38.09	31.48	22.66	14.38	14.96	23.19	32,00	38.28
E ,	31.14	30.91	22,07	13.78	14,30	22.64	31.42	37.73
u	- 3	+ 2	- 2	- 13	- 4	- 5	- 6	+ 5
U '- U ,	+ 0.37	+ 0.01	+ 0.06	+ 0.09	+ 0.08	+ 0.04	0.00	+ 0.03

ß) East clamp; Tagbeobachtungen.

thread distance	2 - 12	5 - 12	6 - 12	10 - 12	14 - 12	18 - 12	19 - 12	22 - 12
M	41	41	40	40	40	40	41	40
A	37.405	31.146	22.314	13.994	14.633	22,938	31.759	38.028
H	0.062	0.077	0.084	0.074	0,080	0.074	0.072	0,069
e '	37.57	31.38	22.54	14.17	14.81	23,21	31.93	38.22
E ,	37.16	30.96	22,03	13.78	14.41	22.73	31.56	37.78
u	- 4	- 3	+5	+1	+2	+2	0	+2
U '- U ,	-0.08	+0.05	- 0.06	- 0.04	+0.05	+0.06	- 0.03	- 0.06

γ) West terminal; Night observations.

thread distance	2 - 12	5 - 12	6 - 12	10 - 12	14 - 12	18 - 12	19 - 12	22 - 12
m	124	124	124	124	124	123	123	123
A	37.453	31.229	22.374	14,050	14.593	22.864	31.713	37.976
H	0,090	0,089	0.085	0,089	0,089	0.083	0.105	0.094
e '	37.92	31.53	22.61	14.33	14.91	23,16	31,99	38.28
E ,	37.13	30.92	22.10	13.75	14,30	22,62	31.41	37.67
U	- 8th	+ 8	+ 2	- 2	+ 2	- 4	0	+ 6
U '- U ,	+ 0.14	- 0,01	- 0.04	- 0.02	+ 0.02	+ 0.05	- 0.03	0.00

δ) West terminal; Tagbeobachtungen.

thread distance	2 - 12	5 - 12	6 - 12	10 - 12	14 - 12	18 - 12	19 - 12	22 - 12
m	50	50	49	50	50	49	50	49
A	37.463	31.234	22.406	14.061	14.528	22.836	31.717	37.944
H	0.087	0.092	0.084	0.092	0.091	0.079	0.104	0.098
e '	37.76	31.45	22,62	14,30	14.82	23.06	32,13	38.28
E ,	37.25	31.04	22,19	13.75	14,30	22.63	31.42	37,70
U	- 5	- 1	+ 2	+10	+ 2	+ 2	+ 1	- 1
U '- U ,	+ 0.08	+ 0.02	0.00	- 0.07	+ 0.06	+ 0.02	+ 0.12	+ 0.09

Here, the *A represent* the sought thread distances by designating as the arithmetic means of the *m* observation values at the same time the most probable values, if the simple GG is to be regarded as correct. These values differ for the different groups, which is not to be expected at first because of the finiteness of the *m* subject to the determination, but also because of the difference between the positions East and West. For in the groups γ and δ the four first distances are consistently larger, the four last ones in the majority of cases smaller than the corresponding distances of the groups α andβ , as is to be assumed in the late fixation of the passage through the central thread in the position terminal West. The corresponding shows the comparison of the above values with those of Dr. Ing. Kobold [1)] values obtained from other observations with greater reliability, which are shown in the following list:

thread distance	2 - 12	5 - 12	6 - 12	10 - 12	14 - 12	18 - 12	19 - 12	22 - 12
A	37^{s}, 443	31^{s}, 195	22^{s}, 355	14^{s}, 030	14^{s}, 591	22^{s}, 893	31^{s}, 735	38^{s}, 006

The ηэσ give the average mean errors as the average of the differences between the observed values and the *A's* . These show only slight fluctuations within the individual groups, according to which the eight error series of each group form a similar system, as was already to be assumed on the basis of their origin. The variation width of the error from the difference between the upper and lower extremes *E* 'and *E ,* can be seen; it is only for the thread distance 2 - 12 of the group α 0^{s} , 95; the magnitude of this value is, however, essentially due to the amount of the upper extreme deviation *U '*= 0^{s} ,66, which significantly exceeds the average expected amount and is considered abnormal.]

[1] [Comp. Annals of the Kaiserl. University Observatory in Strasbourg; I. Bd. 1896. S. XXII: The thread distances and the angle values of the screw.]

[But above all, the values of the *u* and, in connection with it, those of the *U* '- *U* , are of interest because they allow an answer to the question as to whether the asymmetry of the error series is to be regarded as essential or insignificant. Now the *u*- values are consistently very small and have in an unregulated sequence soon positive, now negative sign. The same is to be said of the differences *U* '- *U* , which have no change between the signs only in the group α and here only in the one value 0^s, 37 to a significant height, which, according to the above remarks, can not be considered with regard to the associated upper extreme deviation. From this follows conclusively the conclusion that there is no essential asymmetry. Can be found in the fact that the sign of only 18 among 32 cases thereof confirmation also *u* and *U* '- *U* , are opposite to each other, and thus the reverse law of asymmetry between the difference of the deviation numbers and those of the extreme deviations mar. *A* does not prove itself, while experience shows that it is valid in the presence of predominant essential asymmetry.]

§ 182. [There is therefore no reason to apply the principles of collective asymmetry to the error series. However, in order to show that there are no advantages over the simple law in terms of the correspondence between theory and experience with the application of the two-sided GG, I give below comparison tables in such a form that the empirical values are both those according to simple GG. *A* and the two-sided GG bez. *D* calculated theoretical values are available. The empirical values were obtained from the four groups of eight observation series in such a way that, first of all, in each series of observations, the observed values were determined by their differences with the associated *A*di was replaced by observation errors Δ and then the eight error series of each group were merged into a single series. According to the four groups α , *β* , *γ* , δ , four series of errors were created, which are to be called the series α , *ß*, *γ* , *δ* . The merging of the original series was not subject to any reservations, as they had proved to be similar due to the correspondence between the corresponding average errors η .]

[If you reduce to $i = 0^s$, 05 you get the following results:

III. Reduced distribution tables of error series α , *β* , *γ* , *δ* .

E = 1^s ; *i* = 0.05.

Row α			Row β		
		theor.			theor.

Δ	emp.	Ref. A	bez. D_p	Δ	emp.	bez. A	bez. D_p
- 0.35	-	2.5	2	- 0.30	1	0.5	0.5
- 0.30	6	6.5	5.5	- 0.25	2	2	2
- 0.25	21	17	16	- 0.20	9	8th	8th
- 0.20	38	37	37	- 0.15	21	20.5	20.5
- 0.15	59	69	71	- 0.10	29	40	40.5
- 0.10	108	107	111	- 0.05	**70**	60	60
- 0.05	**154**	139	143	0.00	67	67.5	67.5
0.00	151	152	151.5	+ 0.05	59	58	57.5
+ 0.05	152	140	136	+ 0.10	39	38	38
+ 0.10	100	108	104	+ 0.15	17	19	19
+ 0.15	55	70	68	+0.20	6	7	7
+0.20	36	38.5	38.5	+ 0.25	3	2	2
+ 0.25	18	17.5	18.5	+ 0.30	-	0.5	0.5
+ 0.30	12	7	8th	m =	323	323	323
+ 0.35	3	2	3				
+ 0.40	-	1	1				
+ 0.65	1	-	-				
m =	914	914	914				

Row γ				Row δ			
Δ	theor.			Δ	theor.		
	emp.	Ref. A	bez. D_p		emp.	bez. A	Ref. D_p
- 0.40	-	0.5	0.5	- 0.35	-	1	1
- 0.35	-	2	2	- 0.30	3	3	3
- 0.30	10	6	7	- 0.25	5	7.5	7
- 0.25	19	17	18	- 0.20	15	16	16
- 0.20	42	39	39	- 0.15	29	30	31
- 0.15	69	74	72.5	- 0.10	55	47	47.5

- 0.10	101	117	114	- 0.05	61	61	61.5
- 0.05	159	154.5	151	0.00	64	66	66.5
0.00	**174**	169	169	+ 0.05	**71**	61	60
+ 0.05	163	154.5	158	+ 0.10	44	47	46
+ 0.10	120	117	121	+ 0.15	22	30	30
+ 0.15	73	74	75.5	+0.20	17	16	16
+0.20	37	39	38.5	+ 0.25	4	7.5	7.5
+ 0.25	14	17	16	+ 0.30	5	3	3
+ 0.30	7	6	5	+ 0.35	1	1	1
+ 0.35	0	3	1.5	+ 0.40	1	-	-
+ 0.40	0	0.5	0.5	m =	397	397	397
+ 0.45	1	-	-				
m =	989	989	989				

IV. Elements of error series α , β , γ, δ according to reduced tables.

$\boldsymbol{E} = 1^{s}$.

	α	β	Γ	δ
m	914	323	989	397
A	+ 0.0009	- 0.0025	0.0000	- 0.0004
C	- 0,0015	- 0.0030	+ 0.0022	- 0,0012
D_p	- 0,0111	- 0.0050	+ 0.0094	- 0.0048
D_i	- 0.0281	- 0.0284	+ 0.0038	+ 0.0353
H	.0949	0.0753	0.0923	0.0946
e ,	.0888	0.0741	0.0969	0.0924
e'	0.1008	0.0766	0.0875	.0968
u	- 9	- 8th	+15	- 3
$\boldsymbol{u}$	+ 58	+ 5	- 50	+ 9
p	0.80	0.80	0.77	0.82

In them there is everywhere a so far-reaching agreement between the theoretical values of the symmetrical and the asymmetrical law of distribution, that it seems irrelevant which of them is to be used as a basis.]

[But then the preference of simplicity in favor of the symmetrical law will prevail, whereby it is still significant that one does not have to go back to reduced tables for the calculation of the elements, but the primarily determined average error η or (quadratic) mean error *q* at the Can use distribution statement. In the present case one thus obtains from the primary distribution tables for the η oφ the rows α , *ß, γ, δ* respectively 0^s, 0937; 0^s, 0738; 0^s, 0906; 0^s, 0911, which leads to the following comparison table between theory and experience:

V. Comparison between theory and experience for the simple GG

± *Δ*	*α*		*β*		*Γ*		*δ*	
	emp.	theor.	emp.	theor.	emp.	theor.	emp.	theor.
0.00	151	154	67	69	174	169	64	69
0.05	306	282	129	119	322	309	132	125
0.10	208	216	68	78	221	234	99	94.5
0.15	114	138	38	38	142	148	51	59
0.20	74	74	15	14	79	78	32	30.5
0.25	39	33	5	4	33	34	9	13
0.30	18	12	1	1	17	12	8th	5
0.35	3	4			0	4	1	1
0.40	-	1			0	1	1	-
0.45	-	-			1	-		
0.65	1	-						
m =	914	914	323	323	989	989	397	397

Here the interval denoted by 0.00 would have to be doubled with the limits ± 0.025 in order to be directly comparable with the other intervals, so that of course the theoretical maximum value always falls to the zero value]

[By now, in theory and experience, the two-sided G. G. Although it appears to be applicable but offers no advantage over the simple GG, it will be regarded as a characteristic feature of the error series that its asymmetry is a merely insignificant one, based on unbalanced contingencies. Accordingly, if one were to lay down a criterion for the evaluation of series of errors, one could actually use the asymmetry as such and lay down the principle that series of errors with the characteristics of essential asymmetry should be rejected.]

Attachment.

The *t*- table.

§ 183. [The *t* -table gives the values of G. G., ie of the integral

in their dependence on the argument $t = \Theta : \varepsilon$. Since four-digit integral values generally satisfy the needs of the collective theory of measurement, the four-digit panel, the FIGHTS in WUNDT's Philosophical Studies, in the IX. Vol., Pp. 147-150, has been *reprinted* as *t*- Table I here. However, in order to have another place available for special cases, the five-digit table is also reported as *t*- table II in a corresponding extent.]

[Both tables are based in the same way on the seven-digit plate found in MEYER's lectures on probability calculus pp. 545-549. But since, as usual, the argument values *t are listed* only up to the second decimal, the second differences should generally be consulted for interpolation. In order to avoid this, in the four-digit table in the interval $t = 0$ to t = 1.51, in the five-digit table in the interval $t = 0$ to $t = 2.01$, the argument was continued until the third decimal, so that everywhere sufficient with simple interpolation. For this purpose, at the designated intervals by means of the formula:

due to the seven-digit tabular values, interpolated using their second differences. The third differences could be disregarded.]

[The setup of the tables is modeled on that of the logarithmic tables. In particular, the asterisks found in individual horizontal rows of Plate II have the meaning that the first decimal prefixed to the line should be increased by one.]

The *t*- table I.

t	0	1	2	3	4	5	6	7	8th	9
0.00	0.0000	0011	0023	0034	0045	0056	0068	0079	0090	0102
01	0.0113	0124	0135	0147	0158	0169	0181	0192	0203	0214
02	0.0226	0237	0248	0259	0271	0282	0293	0305	0316	0327
03	0.0338	0350	0361	0372	0384	0395	0406	0417	0429	0440
04	0.0451	0462	0474	0485	0496	0507	0519	0530	0541	0552
05	0.0564	0575	0386	0597	0609	0620	0631	0642	0654	0665

06	0.0676	0687	0699	0710	0721	0732	0744	0755	0766	0777
07	0.0789	0800	0811	0822	0833	0845	0856	0867	0878	0890
08	.0901	0912	0923	0934	0946	0957	0968	0979	0990	1002
09	0.1013	1024	1035	1046	1058	1069	1080	1091	1102	1113
0.10	0.1125	1136	1147	1158	1169	1180	1192	1203	1214	1225
11	.1236	1247	1259	1270	1281	1292	1303	1314	1325	1336
12	.1348	1359	1370	1381	1392	1403	1414	1425	1436	1448
13	0.1459 .	1470	1481	1492	1503	1514	1525	1536	1547	1558
14	.1569	1581	1592	1603	1614	1625	1636	1647	1658	1669
15	.1680	1691	1702	1713	1724	1735	1746	1757	1768	1779
16	.1790	1801	1812	1823	1834	1845	1856	1867	1878	1889
17	.1900	1911	1922	1933	1944	1955	1966	1977	1988	1998
18	.2009	2020	2031	2042	2053	2064	2075	2086	2097	2108
19	0.2118	2129	2140	2151	2162	2173	2184	2194	2205	2216
0.20	.2227	2238	2249	2260	2270	2281	2292	2303	2314	2324
21	.2335	2346	2357	2368	2378	2389	2400	2411	2421	2432
22	.2443	2454	2464	2475	2486	2497	2507	2518	2529	2540
23	.2550	2561	2572	2582	2593	2604	2614	2625	2636	2646
24	.2657	2668	2678	2689	2700	2710	2721	2731	2742	2753
25	.2763	2774	2784	2795	2806	2816	2827	2837	2848	2858
26	.2869	2880	2890	2901	2911	2922	2932	2943	2953	2964
27	.2974	2985	2995	3006	3016	3027	3037	3047	3058	3068
28	.3079	3089	3100	3110	3120	3131	3141	3152	3162	3172
29	.3183	3193	3204	3214	3224	3235	3245	3255	3266	3276
0.30	.3286	3297	3307	3317	3327	3338	3348	3358	3369	3379
31	.3389	3399	3410	3420	3430	3440	3450	3461	3471	3481
32	.3491	3501	3512	3522	3532	3542	3552	3562	3573	3583
33	.3593	3603	3613	3623	3633	3643	3653	3663	3674	3684
34	.3694	3704	3714	3724	3734	3744	3754	3764	3774	3784
35	.3794	3804	3814	3824	3834	3844	3854	3864	3873	3883
36	.3893	3903	3913	3923	3933	3943	3953	3963	3972	3982
37	.3992	4002	4012	4022	4031	4041	4051	4061	4071	4080

38	.4090	4100	4110	4119	4129	4139	4149	4158	4168	4178
39	.4187	4197	4207	4216	4226	4236	4245	4255	4265	4274
0.40	.4284	4294	4303	4313	4322	4332	4341	4351	4361	4370
41	.4380	4389	4399	4408	4418	4427	4437	4446	4456	4465
42	.4475	4484	4494	4503	4512	4522	4531	4541	4550	4559
43	.4569	4578	4588	4597	4606	4616	4625	4634	4644	4653
44	.4662	4672	4681	4690	4699	4709	4718	4727	4736	4746
45	.4755	4764	4773	4782	4792	4801	4810	4819	4828	4837
46	.4847	4856	4865	4874	4883	4892	4901	4910	4919	4928
47	.4937	4946	4956	4965	4974	4983	4992	5001	5010	5019
48	.5027	5036	5045	5054	5063	5072	5081	5090	5099	5108
49	.5117	5126	5134	5143	5152	5161	5170	5179	5187	5196
0.50	.5205	5214	5223	5231	5240	5249	5258	5266	5275	5284
t	0	1	2	3	4	5	6	7	8th	9

The t-table I.

t	0	1	2	3	4	5	6	7	8th	9
0.50	.5205	5214	5223	5231	5240	5249	5258	5266	5275	5284
51	.5292	5301	5310	5318	5327	5336	5344	5353	5362	5370
52	.5379	5388	5396	5405	5413	5422	5430	5439	5448	5456
53	.5465	5473	5482	5490	5499	5507	5516	5524	5533	5541
54	.5549	5558	5566	5575	5583	5591	5600	5608	5617	5625
55	.5633	5642	5650	5658	5667	5675	5683	5691	5700	5708
56	.5716	5724	5733	5741	5749	5757	5765	5774	5782	5790
57	.5798	5806	5814	5823	5831	5839	5847	5855	5863	5871
58	.5879	5887	5895	5903	5911	5919	5927	5935	5943	5951
59	0.5959	5967	5975	5983	5991	5999	6007	6015	6023	6031
0.60	.6039	6046	6054	6062	6070	6078	6086	6093	6101	6109
61	.6117	6125	6132	6140	6148	6156	6163	6171	6179	6186
62	.6194	6202	6209	6217	6225	6232	6240	6248	6255	6263
63	.6270	6278	6286	6293	6301	6308	6316	6323	6331	6338
64	.6346	6353	6361	6368	6376	6383	6391	6398	6405	6413
65	.6420	6428	6435	6442	6450	6457	6464	6472	6479	6486

66	.6494	6501	6508	6516	6523	6530	6537	6545	6552	6559
67	.6566	6573	6581	6588	6595	6602	6609	6616	6624	6631
68	.6638	6645	6652	6659	6666	6673	6680	6687	6694	6701
69	.6708	6715	6722	6729	6736	6743	6750	6757	6764	6771
0.70	.6778	6785	6792	6799	6806	6812	6819	6826	6833	6840
71	0.6847	6853	6860	6867	6874	6881	6887	6894	6901	6908
72	.6914	6921	6928	6934	6941	6948	6954	6961	6968	6974
73	.6981	6988	6994	7001	7007	7014	7021	7027	7034	7040
74	.7047	7053	7060	7066	7073	7079	7086	7092	7099	7105
75	0.7112	7118	7124	7131	7137	7144	7150	7156	7163	7169
76	.7175	7182	7188	7194	7201	7207	7213	7219	7226	7232
77	.7238	7244	7251	7257	7263	7269	7275	7282	7288	7294
78	.7300	7306	7512	7318	7325	7331	7337	7343	7349	7355
79	.7361	7367	7373	7379	7385	7391	7397	7403	7409	7415
0.80	.7421	7427	7433	7439	7445	7451	7457	7462	7468	7474
81	.7480	7486	7492	7498	7503	7509	7515	7521	7527	7532
82	.7538	7544	7550	7555	7561	7567	7572	7578	7584	7590
83	.7595	7601	7607	7612	7618	7623	7629	7635	7640	7646
84	.7651	7657	7663	7668	7674	7679	7685	7690	7696	7701
85	.7707	7712	7718	7723	7729	7734	7739	7745	7750	7756
86	.7761	7766	7772	7777	7782	7788	7793	7798	7804	7809
87	.7814	7820	7825	7830	7835	7841	7846	7851	7856	7862
88	.7867	7872	7877	7882	7888	7893	7898	7903	7908	7913
89	.7918	7924	7929	7934	7939	7944	7949	7954	7959	7964
0.90	.7969	7974	7979	7984	7989	7994	7999	8004	8009	8014
91	.8019	8024	8029	8034	8038	8043	8048	8053	8058	8063
92	.8068	8073	8077	8082	8087	8092	8097	8101	8106	8111
93	.8116	8120	8125	8130	8135	8139	8144	8149	8153	8158
94	.8163	8167	8172	8177	8181	8186	8191	8195	8200	8204
95	.8209	8213	8218	8223	8227	8232	8236	8241	8245	8250
96	.8254	8259	8263	8268	8272	8277	8281	8285	8290	8294
97	.8299	8303	8307	8312	8316	8321	8325	8329	8334	8338
98	.8342	8347	8351	8355	8360	8364	8368	8372	8377	8381

99	.8385	8389	8394	8398	8402	8406	8410	8415	8419	8423
1.00	.8427	8431	8435	8439	8444	8448	8452	8456	8460	8464
t	0	1	2	3	4	5	6	7	8th	9

The t-table I.

t	0	1	2	3	4	5	6	7	8th	9
1.00	.8427	8431	8435	8439	8444	8448	8452	8456	8460	8464
01	.8468	8472	8476	8480	8484	8488	8492	8496	8500	8504
02	.8508	8512	8516	8520	8524	8528	8532	8536	8540	8544
03	.8548	8552	8556	8560	8563	8567	8571	8575	8579	8583
04	.8586	8590	8594	8598	8602	8606	8609	8613	8617	8621
05	.8624	8628	8632	8636	8639	8643	8647	8650	8654	8658
06	.8661	8665	8669	8672	.8676	8680	8683	8687	8691	8694
07	.8698	8701	8705	8708	8712	8716	8719	8723	8726	8730
08	.8733	8737	8740	8744	8747	8751	8754	8758	8761	8765
09	.8768	8771	8775	8778	8782	8785	8789	8792	8795	8799
1.10	.8802	8805	8809	8812	8815	8819	8822	8825	8829	8832
11	.8835	8839	8842	8845	8848	8852	8855	8858	8861	8865
12	.8868	8871	8874	8878	8881	8884	8887	8890	8893	8897
13	.8900	8903	8906	8909	8912	8915	8918	8922	8925	8928
14	.8931	8934	8937	8940	8943	8946	8949	8952	8955	8958
15	.8961	8964	8967	8970	8973	8976	8979	8982	8985	8988
16	.8991	8994	8997	9000	9003	9006	9008	9011	9014	9017
17	0.9020	9023	9026	9029	9031	9034	9037	9040	9043	9046
18	.9048	9051	9054	9057	9060	9062	9065	9068	9071	9073
19	.9076	9079	9082	9084	9087	9090	9092	9095	9098	9100
1.20	.9103	9106	9108	9111	9114	9116	9119	9122	9124	9127
21	.9130	9132	9135	9137	9140	9143	9145	9148	9150	9153
22	.9155	9158	9160	9163	9165	9168	9171	9173	9176	9178
23	.9181	9183	9185	9188	9190	9193	9195	9198	9200	9203
24	0.9205	9207	9210	9212	9215	9217	9219	9222	9224	9227

25	.9229	9231	9234	9236	9238	9241	9243	9245	9248	9250
26	0.9252	9255	9257	9259	9262	9264	9266	9268	9271	9273
27	.9275	9277	9280	9282	9284	9286	9289	9291	9293	9295
28	.9297	9300	9302	9304	9306	9308	9310	9313	9315	9317
29	.9319	9321	9323	9325	9327	9330	9332	9334	9336	9338
1.30	.9340	9342	9344	9346	9348	9350	9352	9355	9357	9359
31	.9361	9363	9365	9367	9369	9371	9373	9375	9377	9379
32	.9381	9383	9385	9387	9389	9390	9392	9394	9396	9398
33	.9400	9402	9404	9406	9408	9410	9412	9413	9415	9417
34	.9419	9421	9423	9425	9427	9428	9430	9432	9434	9436
35	.9438	9439	9441	9443	9445	9447	9448	9450	9452	9454
36	.9456	9457	9459	9461	9463	9464	9466	9468	9470	9471
37	.9473	9475	9477	9478	9480	9482	9483	9485	9487	9488
38	.9490	9492	9494	9495	9497	9499	9500	9502	9503	9505
39	.9507	9508	9510	9512	9513	9515	9516	9518	9520	9521
1.40	.9523	9524	9526	9528	9529	9531	9532	9534	9535	9537
41	.9539	9540	9542	9543	9545	9546	9548	9549	9551	9552
42	.9554	9555	9557	9558	9560	9561	9563	9564	9566	9567
43	.9569	9570	9571	9573	9574	9576	9577	9579	9580	9582
44	.9583	9584	9586	9587	9589	9590	9591	9593	9594	9596
45	.9597	9598	9600	9601	9602	9604	9605	9607	9608	9609
46	.9611	9612	9613	9615	9616	9617	9618	9620	9621	9622
47	0.9624	9625	9626	9628	9629	9630	9631	9633	9634	9635
48	.9637	9638	9639	9640	9642	9643	9644	9645	9647	9648
49	.9649	9650	9651	9653	9654	9655	9656	9657	9659	9660
1.50	.9661	9662	9663	9665	9666	9667	9668	9669	9670	9672
t	0	1	2	3	4	5	6	7	8th	9

The t-table I.

t	0	1	2	3	4	6	6	7	8th	9
1.5	.9661	9673	9684	9695	9706	9716	9726	9736	9745	9755
1.6	.9763	9772	9780	9788	9796	9804	9811	9818	9825	9832
1.7	.9838	9844	9850	9856	9861	9867	9872	9877	9882	9886

1.8	.9891	9895	9899	9903	9907	9911	9915	9918	9922	9925
1.9	.9928	9931	9934	9937	9939	9942	9944	9947	9949	9951
2.0	.9953	9955	9957	9959	9961	9963	9964	9966	9967	9969
2.1	.9970	9972	9973	9974	9975	9976	9977	9979	9980	9980
2.2	.9981	9982	9983	9984	9985	9985	9986	9987	9987	9988
2.3	0.9989	9989	9990	9990	9991	9991	9992	9992	9992	9993
2.4	0.9993	9993	9994	9994	9994	9995	9995	9995	9995	9996
2.5	0.9996	9996	9996	9997	9997	9997	9997	9997	9997	9998
2.6	0.9998	9998	9998	9998	9998	9998	9998	9998	9998	9999
2.7	0.9999	9999	9999	9999	9999	9999	999?	9999	9999	9999
2.8	0.9999	9999	9999	9999	9999	9999	9999	0000	0000	0000

The t-table II.

t	0	1	2	3	4	5	6	7	8th	9
0.00	0.00000	0113	0226	0339	0451	0564	0677	0790	0903	1016
01	1128	1241	1354	1467	1580	1692	1805	1918	2031	2144
02	2256	2369	2482	2595	2708	2820	2933	3046	3159	3271
03	3384	3497	3610	3722	3835	3948	4060	4173	4286	4398
04	4511	4624	4736	4849	4962	5074	5187	5299	5412	5525
05	0,0 5637	5750	5862	5975	6087	6200	6312	6425	6537	6650
06	6762	6875	6987	7099	7212	7324	7436	7549	7661	7773
07	7886	7998	8110	8223	8335	8447	8559	8671	8784	8896
08	0.09008	9120	9232	9344	9456	9568	9680	9792	9904	* 0016
09	0.1 0128	0240	0352	0464	0576	0687	0799	0911	1023	1135
0.10	0.1 1246	1358	1470	1581	1693	1805	1916	2028	2139	2251
11	2362	2474	2585	2697	2808	2919	3031	3142	3253	3365
12	3476	3587	3698	3809	3921	4032	4143	4254	4365	4476
13	4587	4698	4809	4919	5030	5141	5252	5363	5473	5584
14	5695	5805	5916	6027	6137	6248	6358	6468	6579	6689

15	0.1 6800	6910	7020	7130	7241	7351	7461	7571	7681	7791
16	7901	8011	8121	8231	8341	8451	8560	8670	8780	8890
17	0.1 8999	9109	9218	9328	9437	9547	9656	9766	9875	9984
18	0.2 0094	0203	0312	0421	0530	0639	0748	0857	0966	1075
19	1184	1293	1402	1510	1619	1728	1836	1945	2053	2162
0.20	0.2 2270	2379	2487	2595	2704	2812	2920	3028	3136	3244
21	3352	3460	3568	3676	3784	3891	3999	4107	4214	4322
22	4430	4537	4645	4752	4859	4967	5074	5181	5288	5395
23	5502	5609	5716	5823	5930	6037	6144	6250	6357	6463
24	6570	6676	6783	6889	6996	7102	7208	7314	7421	7527
25	0.2 7633	7739	7845	7950	8056	8162	8268	8373	8479	8584
26	8690	8795	8901	9006	9111	9217	9322	9427	9532	9637
27	0.2 9742	9847	9952	* 0056	* 0161	* 0266	* 0370	* 0475	* 0579	* 0684
28	0.3 0788	0892	0997	1101	1205	1309	1413	1517	1621	1725
29	1828	1932	2036	2139	2243	2346	2450	2553	2656	2760
0.30	0,32863	2966	3069	3172	3275	3378	3480	3583	3686	3788
31	3891	3993	4096	4198	4300	4403	4505	4607	4709	4811
32	4913	5014	5116	5218	5319	5421	5523	5624	5725	5827
33	5928	6029	6130	6231	6332	6433	6534	6635	6735	6836
34	6936	7037	7137	7238	7338	7438	7538	7638	7738	7838
35	0.37938	8038	8138	8237	8337	8436	8536	8635	8735	8834
36	8933	9032	9131	9230	9329	9428	9526	9625	9724	9822
37	0.3 9921	* 0019	* 0117	* 0215	* 0314	* 0412	* 0510	* 0608	* 0705	* 0803
38	0.4 0901	0999	1096	1194	1291	1388	1486	1583	1680	1777
39	1874	1971	2068	2164	2261	2357	2454	2550	2647	2743
0.40	0.4 2839	2935	3031	3127	3223	3319	3415	3510	3606	3701

41	3797	3892	3988	4083	4178	4273	4368	4463	4557	4652
42	4747	4841	4936	5030	5124	5219	5313	5407	5501	5595
43	5689	5782	5876	5970	6063	6157	6250	6343	6436	6529
44	6623	6715	6808	6901	6994	7086	7179	7271	7364	7456
45	0.4 7548	7640	7732	7824	7916	8008	8100	8191	8283	8374
46	8466	8557	8648	8739	8830	8921	9012	9103	9193	9284
47	0.4 9375	9465	9555	9646	9736	9826	9916	* 0006	* 0096	* 0185
48	0.5 0275	0365	0454	0543	0633	0722	0811	0900	0989	1078
49	1167	1256	1344	1433	1521	1610	1698	1786	1874	1962
0.50	0.5 2050	2138	2226	2313	2401	2488	2576	2663	2750	2837
t	0	1	2	3	4	5	6	7	8th	9

The t-table II.

t	0	1	2	3	4	5	6	7	8th	9
0.50	0.5 2050	2138	2226	2313	2401	2488	2576	2663	2750	2837
51	2924	3011	3098	3185	3272	3358	3445	3531	3617	3704
52	3790	3876	3962	4048	4134	4219	4305	4390	4476	4561
53	4646	4732	4817	4902	4987	5071	5156	5241	5325	5410
54	5494	5578	5662	5746	5830	5914	5998	6082	.6165	6249
55	0.5 6332	6416	6499	6582	6665	6748	6831	6914	6996	7079
56	7162	7244	7326	7409	7491	7573	7655	7737	7818	7900
57	7982	8063	8144	8226	8307	8388	8469	8550	8631	8712
58	8792	8873	8953	9034	9114	9194	9274	9354	9434	9514
59	0.5 9594	9673	9753	9832	9912	9991	* 0070	* 0149	* 0228	* 0307
0.60	0.6 0386	0464	0543	0621	0700	0778	0856	0934	1012	1090

61	1168	1246	1323	1401	1478	1556	1633	1710	1787	1864
62	1941	2018	2095	2171	2248	2324	2400	2477	2553	2629
63	2705	2780	2856	2932	3007	3083	3158	3233	3309	3384
64	3459	3533	3608	3683	3757	3832	3906	3981	4055	4129
65	0.6 4203	4277	4351	4424	4498	4571	4645	4718	4791	4865
66	4938	5011	5083	5156	5229	5301	5374	5446	5519	5591
67	5663	5735	5807	5878	5950	6022	6093	6165	6236	6307
68	6378	6449	6520	6591	6662	6732	6803	6873	6944	7014
69	7084	7154	7224	7294	7364	7433	7503	7572	7642	7711
0.70	0.6 7780	7849	7918	7987	8056	8125	8193	8262	8330	8398
71	8467	8535	8603	8671	8738	8806	8874	8941	9009	9076
72	9143	9210	9277	9344	9411	9478	9545	9611	9678	9744
73	0.6 9810	9877	9943	* 0009	* 0075	* 0140	* 0206	* 0272	* 0337	* 0402
74	0.7 0468	0533	0598	0663	0728	0793	0858	0922	0987	1051
75	0.7 1116	1180	1244	1308	1372	1436	1500	1563	1627	1690
76	1754	1817	1880	1943	2006	2069	2132	2195	2257	2320
77	2382	2444	2507	2569	2631	2693	2755	2816	2878	2940
78	3001	3062	3124	3185	3246	3307	3368	3429	3489	3550
79	3610	3671	3731	3791	3851	3911	3971	4031	4091	4151
0.80	0.7 4210	4270	4329	4388	4447	4506	4565	4624	4683	4742
81	4800	4859	4917	4976	5034	5092	5150	5208	5266	5323
82	5381	5439	5496	5553	5611	5668	5725	5782	5839	5896
83	5952	6009	6066	6122	6178	6234	6291	6347	6403	6459
84	6514	6570	6626	6681	6736	6792	6847	6902	6957	7012
85	0,77067	7122	7176	7231	7285	7340	7394	7448	7502	7556
86	7610	7664	7718	7771	7825	7878	7932	7985	8038	8091
87	8144	8197	8250	8302	8355	8408	8460	8512	8565	8617
88	8669	8721	8773	8824	8876	8928	8979	9031	9082	9133
89	9184	9235	9286	9337	9388	9439	9489	9540	9590	9641

0.90	0.7 9691	9741	9791	9841	9891	9941	9990	* 0040	* 0090	* 0139
91	0.8 0188	0238	0287	0336	0385	0434	0482	0531	0580	0628
92	0677	0725	0773	0822	0870	0918	0966	1013	1061	1109
93	1156	1204	1251	1298	1346	1393	1440	1487	1534	1580
94	1627	1674	1720	1767	1813	1859	1905	1951	1997	2043
93	0.8 2089	2135	2180	2226	2271	2317	2362	2407	2452	2497
96	2542	2587	2632	2677	2721	2766	28 10	2855	2899	2943
97	2987	3031	3075	3119	3162	3206	3250	3293	3337	3380
98	3423	3466	3509	3552	3595	3638	3681	3723	3766	3808
99	3851	3893	3935	3977	4020	4061	4103	4145	4187	4229
1.00	0.8 4270	4312	4353	4394	4435	4477	4518	4559	4600	4640
t	0	1	2	3	4	5	6	7	8th	9

The t-table II.

t	0	1	2	3	4	5	6	7	8th	9
1.00	0.8 4270	4312	4353	4394	4435	4477	4518	4559	4600	4640
01	4681	4722	4762	4803	4843	4883	4924	4964	5004	5044
02	5084	5124	5163	5203	5243	5282	5322	5361	5400	5439
03	5478	5517	5556	5595	5634	5673	5711	5750	5788	5827
04	5865	5903	5941	5979	6017	6055	6093	6131	6169	6206
05	0.8 6244	6281	6318	6356	6393	6430	6467	6504	6541	6578
06	6614	6651	6688	6724	6760	6797	6833	6869	6905	6941
07	6977	7013	7049	7085	7120	7156	7191	7227	7262	7297
08	7333	7368	7403	7438	7473	7507	7542	7577	7611	7646
09	7680	7715	7749	7783	7817	7851	7885	7919	7953	7987
1.10	0.8 8021	8054	8088	8121	8155	8188	8221	8254	8287	8320
11	8353	8386	8419	8452	8484	8517	8549	8582	8614	8647
12	8679	8711	8743	8775	8807	8839	8871	8902	8934	8966

13	8997	9029	9060	9091	9122	9154	9185	9216	9247	9277
14	9308	9339	9370	9400	9431	9461	9492	9522	9552	9582
15	0.8 9612	9642	9672	9702	9732	9762	9792	9821	9851	9880
16	0.8 9910	9939	9968	9997	* 0027	* 0056	* 0085	* 0114	* 0142	* 0171
17	0.9 0200	0229	0257	0286	0314	0343	0371	0399	0428	0456
18	0484	0512	0540	0568	0595	0623	0651	0678	0706	0733
19	0761	0788	0815	0843	0870	0897	0924	0951	0978	1005
1.20	0.9 1031	1058	1085	1111	1138	1164	1191	1217	1243	1269
21	1296	1322	1348	1374	1399	1425	1451	1477	1502	1528
22	1553	1579	1604	1630	1655	1680	1705	1730	1755	1780
23	1805	1830	1855	1879	1904	1929	1953	1978	2002	2026
24	2051	2075	2099	2123	2147	2171	2195	2219	2243	2266
25	0.9 2290	2314	2337	2361	2384	2408	2431	2454	2477	2500
26	2524	2547	2570	2593	2615	2638	2661	2684	2706	2729
27	2751	2774	2796	2819	2841	2863	2885	2907	2929	2951
38	2973	2995	3017	3039	3061	3082	3104	3126	3147	3168
29	3190	3211	3232	3254	3275	3296	3317	3338	3359	3380
1.30	0.9 3401	3422	3442	3463	3484	3504	3525	3545	3566	3586
31	3606	3627	3647	3667	3687	3707	3727	3747	3767	3787
32	3807	3826	3846	3866	3885	3905	3924	3944	3963	3982
33	4002	. 4021	4040	4059	4078	4097	4116	4135	4154	4173
34	4191	4210	4229	4247	4266	4284	4303	4321	4340	4358
35	0.9 4376	4394	4413	4431	4449	4467	4485	4503	4521	4538
36	4556	4574	4592	4609	4627	4644	4662	4679	4697	4714
37	4731	4748	4766	4783	4800	4817	4834	4851	4868	4885
38	4902	4918	4935	4952	4968	4985	5002	5018	5035	5051
39	5067	5084	5100	5116	5132	5148	5165	5181	5197	5213

1.40	0.9 5229	5244	5260	5276	5292	5307	5323	5339	5354	5370
41	5385	5401	5416	5431	5447	5462	5477	5492	5507	5323
42	5538	5553	5568	5582	5597	5612	5627	5642	5656	5671
43	5686	5700	5715	5729	5744	5758	5773	5787	5801	5815
44	5830	5844	5858	5872	5886	5900	5914	5928	5942	5956
45	0.9 5970	5983	5997	6011	6024	6038	6051	6065	6078	6092
46	6105	6119	6132	6145	6159	6172	6185	6198	6211	6224
47	6237	6250	6263	6276	6289	6302	6315	6327	6340	6353
48	6365	6378	6391	6403	6416	6428	6440	6453	6465	6478
49	6490	6502	6514	6526	6539	6551	6563	6575	6587	6599
1.50	0.9 6611	6622	6634	6646	6658	6670	6681	6693	6705	6716
t	0	1	2	3	4	5	6	7	8th	9

The t-table II.

t	0	1	2	3	4	5	6	7	8th	9
1.50	0.9 6611	6622	6634	6646	6658	6670	6681	6693	6705	6716
51	6728	6739	6751	6762	6774	6785	6796	6808	6819	6830
52	6841	6853	6864	6875	6886	6897	6908	6919	6930	6941
53	6952	6962	6973	6984	6995	7006	7016	7027	7037	7048
54	7059	7069	7080	7090	7100	7111	7121	7131	7142	7152
55	0.97162	7172	7183	7193	7203	7213	7223	7233	7243	7253
56	7263	7273	7283	7292	7302	7312	7322	7331	7341	7351
57	7360	7370	7379	7389	7398	7408	7417	7427	7436	7445
58	7455	7464	7473	7482	7492	7501	7510	7519	7528	7537
59	7546	7555	7564	7573	7582	7591	7600	7609	7617	7626
1.60	0.9 7635	7644	7652	7661	7670	7678	7687	7695	7704	7712
61	7721	7729	7738	7746	7754	7763	7771	7779	7787	7796
62	7804	7812	7820	7828	7836	7844	7852	7860	7868	7876
63	7884	7892	7900	7908	7916	7924	7931	7939	7947	7955
64	7962	7970	7977	7985	7993	8000	8008	8015	8023	8030

65	0.98038	8045	8052	8060	8067	8074	8082	8089	8096	8103
66	8110	8118	8125	8132	8139	8146	8153	8160	8167	8174
67	8181	8188	8195	8202	8209	8215	8222	8229	8236	8243
68	8249	8256	8263	8269	8276	8283	8289	8296	8302	8309
69	8315	8322	8328	8335	8341	8347	8354	8360	8366	8373
1.70	0.9 8379	8385	8392	8398	8404	8410	8416	8422	8429	8435
71	8441	8447	8453	8459	8465	8471	8477	8483	8489	8494
73	8500	8506	8512	8518	8524	8529	8535	8541	8546	8552
73	8558	8563	8569	8575	8580	8586	8591	8597	8602	8608
74	8613	8619	8624	8630	8635	8641	8646	8651	8657	8662
75	0.9 8667	8672	8678	8683	8688	8693	8699	8704	8709	8714
76	8719	8724	8729	8734	8739	8744	8749	8754	8759	8764
77	8769	8774	8779	8784	8789	8793	8798	8803	8808	8813
78	8817	8822	8827	8832	8836	8841	8846	8850	8855	8859
79	8864	8869	8873	8878	8882	8887	8891	8896	8900	8905
1.80	0.9 8909	8913	8918	8922	8927	8931	8935	8940	8944	8948
81	8952	8957	8961	8965	8969	8974	8978	8982	8986	8990
82	8994	8998	9002	9007	9011	9015	9019	9023	9027	9031
83	9035	9039	9043	9046	9050	9054	9058	9062	9066	9070
84	9074	9077	9081	9085	9089	9093	9096	9100	9104	9107
85	0.99111	9115	9118	9122	9126	9129	9133	9137	9140	9144
86	9147	9151	9154	9158	9161	9165	9168	9172	9175	9179
87	9183	9185	9189	9192	9196	9199	9202	9206	9209	9212
88	9216	9219	9222	9225	9229	9232	9235	9238	9242	9245
89	9248	9251	9254	9257	9261	9264	9267	9270	9273	9276
1.90	0.9 9279	9282	9285	9288	9291	9294	9297	9300	9303	9306
91	9309	9312	9315	9318	9321	9324	9326	9329	9332	9335
92	9338	9341	9343	9346	9349	9352	9355	9357	9360	9363
93	9366	9368	9371	9374	9376	9379	9382	9384	9387	9390
94	9392	9395	9397	9400	9403	9405	9408	9410	9413	9415
95	0.9 9418	9420	9423	9425	9428	9430	9433	9435	9438	9440
96	9443	9445	9447	9450	9452	9455	9457	9459	9462	9464
97	9466	9469	9471	9473	9476	9478	9480	9482	9485	9487

98	9489	9491	9494	9496	9498	9500	9502	9505	9507	9509
99	9511	9513	9515	9518	9520	9522	9524	9526	9528	9530
2.00	0.99532	9534	9536	9538	9540	9542	9544	9546	9548	9550
t	0	1	2	3	4	5	6	7	8th	9

The t-table II.

t	0	1	2	3	4	5	6	7	8th	9
2.0	0.99532	9552	9572	9591	9609	9626	9642	9658	9673	9688
2.1	9702	9715	9728	9741	9753	9764	9775	9785	9795	9805
2.2	9814	9822	9831	9839	9846	9854	9861	9867	9874	9880
2.3	9886	9891	9897	9902	9906	9911	9915	9920	9924	9928
2.4	9931	9935	9938	9941	9944	9947	9950	9952	9955	9957
2.5	0.9 9959	9961	9963	9965	9967	9969	9971	9972	9974	9975
2.6	9976	9978	9979	9980	9981	9982	9983	9984	9985	9986
2.7	9987	9987	9988	9989	9989	9990	9991	9991	9992	9992
2.8	9992	9993	9993	9994	9994	9994	9995	9995	9995	9996
2.9	9996	9996	9996	9997	9997	9997	9997	9997	9997	9998
3.0	0.9 9998	9998	9998	9998	9998	9998	9998	9999	9999	9999
3.1	9999	9999	9999	9999	9999	9999	9999	9999	9999	9999
3.2	0.9 9999	9999	9999	* 0000	* 0000	* 0000	* 0000	* 0000	* 0000	* 0000