ROTATING UNIVERSES IN GENERAL RELATIVITY THEORY

KURT GÖDEL
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In this lecture I am setting forth the main results (for the most part without proofs) to which my investigations on rotating universes have led me so far.

1 Definition of the type of rotatory solutions to be considered

I am starting from the relativistic field equations: [1]

$$R_{ik} - \frac{1}{2} g_{ik} R = T_{ik} - \lambda g_{ik} \tag{1.1}$$

and I am assuming that:

1. the relative velocity of masses (i.e. galactic systems) close to each other is small compared with c.
2. no other forces except gravitation come into play.

Under these assumptions T_{ik} takes on the form:

$$T_{ik} = \rho v_i v_k \tag{1.2}$$

where

$$\rho > 0, \tag{1.3}$$

$$g^{ik} v_i v_k = -1 \tag{1.4}$$

and, of course:

$$\text{The signature of } g_{ik} \text{ is } +2. \tag{1.5}$$

The local angular velocity of matter relative to the compass of inertia can be represented by the following vector ω (which is always orthogonal on v):

$$\omega^i = \frac{\epsilon^{iklm}}{12(-g)^{1/2}} a_{klm} \tag{1.6}$$

where the skew-symmetric tensor a_{klm} is defined by:

$$a_{klm} = v_k \cdot \left(\frac{\partial v_l}{\partial x_m} - \frac{\partial v_m}{\partial x_l} \right) + v_l \cdot \left(\frac{\partial v_m}{\partial x_k} - \frac{\partial v_k}{\partial x_m} \right) + v_m \cdot \left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k} \right) \quad . \tag{1.7}$$

That ω represents the angular velocity relative to the compass of inertia is seen as follows: In a coordinate system which, in its origin, is geodesic and normal, and in whose origin

[1] I am supposing that such measuring units are introduced as make $c = 1$, $8\pi\kappa/c^2 = 1$.

matter is at rest (i.e. for which in $O : \partial g_{ik}/\partial x_l$, $g_{ik} = \eta_{ik}$, $v^4 = 1$, $v^i = 0$ for $i \neq 4$) [2], one obtains for ω^i in O:

$$\omega^1 = \frac{1}{2} \cdot \left(\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3} \right) = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x_2} \left(\frac{v^3}{v^4} \right) - \frac{\partial}{\partial x_3} \left(\frac{v^2}{v^4} \right) \right) \ , \ \text{etc.} \tag{1.8}$$

$$\omega^4 = 0 \ . \tag{1.9}$$

In such a coordinate system, however, since parallel-displacement (in its origin) means constancy of the components, the angular velocity relative to the compass of inertia, in O, is given by the same expressions as in Newtonian physics, i.e. the right-hand sides of equation (1.8) are its components. *Evidently ω is the only vector the first 3 components of which, in the particular coordinate systems defined, coincide with the angular velocity computed as in Newtonian physics and the 4th component is 0.*
Any Riemann 4-space with some ρ, v_i defined in it, which everywhere satisfies the conditions (1.1)-(1.5) and permits of no extension free from singularities, and for which, moreover, ω is continuous and $\neq 0$ in every point, represents a rotating universe. However, in the sequel I am chiefly concerned with solutions satisfying the following three further postulates (suggested both by observation and theory):

I. The solution is to be homogeneous in space (i.e. for any two world lines of matter l, m there is to exist a transformation of the solution into itself which carries l into m).

II. Space is to be finite (i.e. the topological space whose points are the world lines of matter is to be closed, i.e. compact).

III. ρ is not to be a constant.

Postulate III is indispensable also for rotating universes, since it can be proved that a *red-shift which, for small distances, increases linearly with the distance implies an expansion, no matter whether the universe rotates or not.* [3]
As to the question of the existence of rotating solutions satisfying the postulates I, II, III, cf. §5.

[2] A coordinate system satisfying the first two conditions may fittingly be called a "local inertial system".

[3] Provided, of course, that the atomic constants do not vary in time and space, or, to be more exact, provided that the dimensionless numbers definable in terms of the constants of nature (such as e^2/hc) are the same everywhere.

2 Some general properties of these solutions

In view of III the equation $\rho =$ const. defines a one-parameter system of 3-spaces. *In rotating universes these 3-spaces of constant density cannot be orthogonal on the world lines of matter*. This follows immediately from the fact that $a_{klm} = 0$ is the necessary and sufficient condition for the existence of any system of 3-spaces orthogonal on a vector field v.
The inclination of the world lines of matter toward the spaces of constant density yields *a directly observable necessary and sufficient criterion for the rotation of an expanding spatially homogeneous and finite universe: namely, for sufficiently great distances, there must be more galaxies in one half of the sky than in the other half.*
In the first approximation, i.e., for solutions differing little from one spatially isotropic, the magnitude of this effect is given by the following theorem: *If N_1, N_2 are the numbers of galaxies in the two hemispheres into which a spatial sphere*[1] *of radius r (small compared with the world radius R) is decomposed by a plane orthogonal on ω, then*:

$$\frac{|N_1 - N_2|}{N_1 + N_2} = \frac{9}{8} \cdot \frac{|\omega|\, r\, R\, h}{c^2} \tag{2.1}$$

where h is Hubble's constant ($= \dot{R}/R$).
For plausible values of the constants (where ω is estimated from the velocity of rotation of the galaxies[2]) this effect is extremely small. But the uncertainty in the knowledge of the constants is too great for drawing any definitive conclusions.
The group of transformations existing owing to I evidently carries each of the spaces $\rho =$ const. into itself, and therefore (the case of isotropy being excluded) can only have 3 or 4 parameters[3]. The number 4 (i.e., the case of rotational symmetry) cannot occur either. There exist no rotationally symmetric rotating universes satisfying the conditions stated in §1[4]. The only symmetry around one point which can occur is that of one rotation by π. This case will be referred to as the symmetric one.
In any case the group of transformations must be 3-parameter. Since moreover, owing to II, it must be compact, and since (as can easily be shown) it cannot be commutative in rotating universes[5]. Hence also the topological connectivity of space must be that of

1 I.e., one situated in a 3-space orthogonal on v at the point under consideration.
2 Cf. my paper in Reviews of Modern Physics vol. 21 (1949) p. 450.
3 There exists, in every space $\rho =$ const., a positive definite metric which is carried into itself, namely the metric h_{ik} defined below.
4 This even is true irrespective of postulate II (the finiteness of space).
5 The reason is that the transformations of any rotating solution of the type characterized in §1 must be isomorphic (as a group of transformations) with the right (or the left) translations of a 3-space

a spherical or elliptical 3-space.
The metric g_{ik} can be decomposed (relative to the world lines of matter) into a space-metric $\overline{g_{ik}}$ and a time-metric $\overline{\overline{g_{ik}}}$, by defining the spatial distance of two neighbouring points P_1, P_2 to be the orthogonal distance of the two world lines of matter passing through P_1, P_2, and the temporal distance to be the orthogonal projection of $P_1 P_2$ on one of these two lines. This decomposition evidently is exactly that which (in the small) holds for the observers moving along the world lines of matter. It has the following properties:

$$\overline{\overline{g_{ik}}} = -\, v_i v_k \,, \quad \overline{g_{ik}} = g_{ik} + v_i v_k \,, \tag{2.2}$$

$$\mathrm{DET}\,(\overline{g_{ik}}) = \mathrm{DET}\,(\overline{\overline{g_{ik}}}) = 0 \;. \tag{2.3}$$

If the coordinate system is so chosen that the x_4-lines are the world lines of matter and the x_4-coordinate measures the length of these lines, $\overline{g_{ik}}$ takes on the form:

$$\overline{g_{ik}} = \begin{Vmatrix} h_{ik} & 0 \\ 0 & 0 \end{Vmatrix} \tag{2.4}$$

(where h_{ik} is positive definite) and the Hubble-constant in the space-direction $\mathrm{d}\,x^i$ (orthogonal on v), as measured by an observer moving along with matter, becomes equal to:

$$\frac{1}{2}\,\frac{\dot{h}_{ik}\,\mathrm{d}\,x^i\,\mathrm{d}\,x^k}{h_{ik}\,\mathrm{d}\,x^i\,\mathrm{d}\,x^k}\,, \text{where } \dot{h}_{ik} = \frac{\partial\,h_{ik}}{\partial\,x_4} \;. \tag{2.5}$$

The surface $\dot{h}_{ik}\,x_i\,x_k$ in the 3-dimensional subspace, orthogonal on v, of the tangent space, may be called the ellipsoid of expansion or, more generally, the quadric of expansion.
The theorem about the nonexistence of rotationally symmetric solutions [6], *under the additional hypothesis that the universe contains no closed time-like lines* (cf. §3), can be strengthened to the statement that *the quadric of expansion, at no moment of time, can be rotationally symmetric around* ω. In particular it can never be a sphere, i.e., the expansion is necessarily coupled with a deformation. This even is true for all solutions satisfying I-III and gives another directly observable property of the rotating universes of this type.
Moreover the asymmetry of the expansion around ω opens up a possibility for the explanation of the spiral structure of the galaxies. For, if under these circumstances a condensation is formed, the chances are that it will become an oblong body rotating around one of its smaller axes; and such a body, because its outer parts will rotate more slowly, will, in the course of time, be bent into a spiral. It remains to be seen whether a quantitative elaboration of this theory of the formation of spirals will lead to agreement with observation.

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of constant positive curvature, or with these translations plus certain rotations by an angle π

[6] This theorem makes it very likely that there exist no rotating spatially homogeneous and expanding solutions whatsoever in which the ellipsoid of expansion is permanently rotationally symmetric around ω.

3 Rotation and time-metric

The formulae (1.6), (1.7), (2.2) show that it is, in the first place, the time-metric (relative to the observers moving along with matter) which determines the behaviour of the compass of inertia. In fact *a necessary and sufficient condition for a spatially homogeneous universe to rotate is that the local simultaneity of the observers moving along with matter be not integrable* (i.e., do not define a simultaneity in the large). This property of the time-metric in rotating universes is closely connected with the possibility of closed time-like lines.

The latter anomaly, however, occurs only if the angular velocity surpasses a certain limit. This limit, roughly speaking, is that value of $|\omega|$ for which the maximum linear velocity caused by the rotation becomes equal to c; i.e., it is approximately c/R if, at the moment considered, the space-metric in the 3-space $\rho =$ const. does not differ too much from a space of the constant curvature $1/R^2$. *The precise necessary and sufficient condition for the nonexistence of closed time-like lines* (provided that the one-parameter manifold of the spaces $\rho =$ const. is not closed) *is that the metric in the spaces of constant density be space-like* [1]. This holds for solutions satisfying all conditions stated in §1.

For these solutions, also, *the nonexistence of closed time-like lines is equivalent with the existence of a "world-time"*, where by a world-time we mean an assignment of a real number t to every space-time point, which has the property that t always increases if one moves along a time-like line in its positive direction [2]. If in addition any two 3-spaces of simultaneity are equidistant and the difference of t is their distance, one may call it a *metric world-time*. If the spaces of constant density are space-like, a metric world-time can be defined by taking these 3-spaces as spaces of simultaneity. Evidently (up to transformations $\bar{t} = f(t)$) this is the only world-time invariant under the group of transformations of the solution.

[1] This condition, too, means that at the border separating the two cases the linear velocity caused by the rotation becomes equal to c, if by this linear velocity is understood the velocity of matter relative to the orthogonals (i.e. all parallel lines) on the spaces of constant density.

[2] A time-like vector is positive if it is contained in the same half of the light-cone as the vector v.

4 Behaviour of the angular velocity in the course of the expansion

No matter whether postulates I-III are satisfied or not, the temporal change of ω is described by the following theorem: *In a coordinate system in which the x_4-lines are the world lines of matter, $g_{44} = -1$ everywhere, and moreover $g_{i4} = 0$ (for $i \neq 4$) on the X_4-axis, one has along the whole X_4-axis*:

$$\omega^i (-g)^{1/2} = \omega^i h^{1/2} = \text{const.} \qquad (i = 1, 2, 3) \ . \tag{4.1}$$

The proof can be given in a few lines: Evidently $v^4 = 1$, $v^i = 0$ (for $i \neq 4$) everywhere; hence: $v_i = g_{i4}$. Substituting these values of v_i in equation (1.7), one obtains on X_4:

$$a_{4ik} = \frac{\partial g_{4k}}{\partial x_i} - \frac{\partial g_{4i}}{\partial x_k} , \qquad a_{123} = 0 \ . \tag{4.2}$$

But $\partial g_{4i}/\partial x_4 = 0$ (because the x_4-lines are geodesics and $g_{44} = -1$). Hence by equation (4.2), $\partial a_{klm}/\partial x_4 = 0$ on X_4. Hence by equation (1.6) also, $\partial (\omega^i (-g)^{1/2})/\partial x_4 = 0$ on X_4.
The equation (4.1) means two things:

A. that the vector ω (or, to be more exact, the lines l_ω whose tangent everywhere has the direction ω) permanently connect the same particles with each other,

B. that the absolute value $|\omega|$ increases or decreases in proportion to the contraction or expansion of matter orthogonal on ω, where this contraction or expansion is measured by the area of the intersection of an infinitesimal spatial cylinder [1] around l_ω (permanently including the same particles) with a surface orthogonal on l_ω.

Since in the proof of equation (4.1), nothing was used except the fact that the world lines of matter are geodesics (and in particular the homogeneity of space was not used), (4.1), and therefore A, B, also describe the behaviour of the angular velocity, if condensations are formed under the influence of gravitation [2]; i.e., $|\omega|$, under these circumstances, increases by the same law as in Newtonian mechanics.
The direction of ω, even in a homogeneous universe, need not be displaced parallel to itself along the world lines of matter. *The necessary and sufficient condition for it to be*

[1] cf. footnote 1, chapter 2
[2] Of course, only as long as the gas and radiation pressure remain small enough to be neglected.

displaced parallel at a certain moment is that it coincides with one of the principal axes of the quadric of expansion. For, if P, Q are two neighbouring particles connected by ω, then, only under the condition just formulated, the direction PQ at the given moment, will be at rest relative to the compass of inertia (in order to see this one only has to introduce the local inertial system defined in §1 (cf. footnote 2, chapter 1) and then argue exactly as in Newtonian physics). Since however (because of A) the direction of ω coincides *permanently* with the direction of PQ, the same condition applies for the direction of ω. This condition however, in general, is not satisfied (only in the symmetric case it is always satisfied).

The fact that the direction of ω need not be displaced parallel to itself might be the reason for the irregular distribution of the directions of the axes of rotation of the galaxies (which at first sight seems to contradict an explanation of the rotation of the galaxies from a rotation of the universe). For, if the axis of rotation of the universe is not displaced parallel, the direction of the angular momentum of a galaxy will depend on the moment of time at which it was formed.

5 Existence theorems

It can be shown that, *for any value of λ (including 0), there exist ∞*[1] *rotating solutions satisfying all conditions stated in §1. The same is true if in addition it is required that a world-time should exist (or should not exist).* The value of the angular velocity is quite arbitrary, even if ρ and the mean world radius (at the moment under consideration) are given. In particular, there are existing rotating solutions with $\lambda = 0$ which differ arbitrarily little from the spatially isotropic solution with $\lambda = 0$.

Thus the problem arises of distinguishing, by properties of symmetry or simplicity, certain solutions in this vast manifold of solutions. E.g., one might try to require that the universe should expand from one point and contract to one point.

[1] cf. footnote 5, chapter 2

6 Method of proof

The method of proof by which the results given above were obtained is based on postulate I of §1. This postulate implies that all world lines of matter (and all orthogonals on the spaces of constant density) are equivalent with each other. It is, therefore, sufficient to confine the consideration to one such world line (or one such orthogonal). This reduces the problem to a system of ordinary differential equations.
Moreover, this system of differential equations can be derived from a Hamiltonian principle, i.e., it is a problem of analytical mechanics with a finite number of degrees of freedom. The equations of relativity theory, however, assign definite values to the integrals of energy and momentum, so that the relativistic problem is a little more special than the corresponding one of analytical mechanics. The symmetric case, by means of the integrals of momentum, can be reduced to a problem with three degrees of freedom (g_1, g_2, g_3), whose Lagrangian function reads as follows:

$$\left\{\sum_{i<k} \frac{\dot{g}_i\,\dot{g}_k}{g_i\,g_k} + \frac{1}{g}\cdot\left[2\sum_i g_i^2 - \left(\sum_i g_i\right)^2\right] + \frac{V^2}{g_1\,(g_2-g_3)^2}\right\} g^{1/2} + 2\cdot\left(1+\frac{V^2}{g_1}\right)^{1/2} \tag{6.1}$$

where $g = g_1\,g_2\,g_3$ and V is a constant which determines the velocity of rotation. The general case can be reduced to a system of differential equations of the 4th order.

7 Stationary rotating solutions

It might be suspected that the desired particular solutions (cf. §5 above) will have a close relationship to the stationary homogeneous solutions, and it is therefore of interest to investigate these, too. By a stationary homogeneous solution we mean one whose group, for any two points P, Q of the whole 4-space, contains transformations carrying P into Q. These solutions can all be determined and expressed by elementary functions. One thus obtains the following results:

A. *There exists no stationary homogeneous solutions with* $\lambda = 0$.

B. *There exist rotating stationary homogeneous solutions with finite space, no closed time-like lines, and* $\lambda > 0$; in particular also such as differ arbitrarily little from Einstein's static universe.

The world lines of matter in these solutions, however, are not equidistant: neighbouring particles of matter, relative to the compass of inertia, rotate around each other, not in circles, but in ellipses (or, to be more exact, in rotating ellipses).

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N.J., U.S.A.